The Black Hole Information Paradox
Assignment 3

Due Saturday, 20 February 2021

1. **Haar measure on pure states**
   In the lectures, we introduced a Haar measure on pure states. Given a state of the form
   $$|\Psi\rangle = \sum_{i=1}^{W} a_i |E_i\rangle,$$
   The measure was
   $$d\mu_\Psi = \frac{1}{V}\delta(\sum_i |a_i|^2 - 1)\prod_i d^2 a_i.$$
   a) Find V so that the measure is correctly normalized: $\int d\mu_\Psi = 1$. Remember that a_i are complex numbers.
   b) Find the value of
   $$\int a_i a_j^* a_k a_l^* d\mu_\Psi.$$

2. **Simulating random pure states**
   Consider a system of n qubits, so that the total number of states is 2^n. Consider the operator
   $$O = \frac{1}{n}\sum \sigma_i^z,$$
   where σ_i^z is the σ^z matrix acting on qubit i. This operator measures the "average" value of the z-spin in the system. For $n = 15$, write a computer program to sample random pure states in this Hilbert space and plot the probability distribution that you obtain for values of O. Check that its width is proportional to $2^{-\frac{n}{2}}$.

3. **Canonical states**
   In the lectures, we considered a process of picking states from an energy band with uniform probability. Now consider the ensemble of states
   $$|\Psi\rangle = \frac{1}{\sqrt{Z(\beta)}}\sum_i e^{-\frac{\beta E_i}{2}} e^{i\theta_i}|E_i\rangle,$$
   where $Z(\beta)$ is the partition function; θ_i are angles picked with uniform probability from $[0, 2\pi]$ and the sum runs over all energy eigenstates.

   Show that the expectation value of an operator in such states is the same as its expectation value in the *canonical ensemble.* Write down an expression for its fluctuations away from the canonical expectation value, as we did for states that mimicked the microcanonical ensemble.

4. **CHSH correlators**
Consider the following state of two qubits.

$$|\Psi\rangle = a|00\rangle + b|11\rangle,$$

where a, b are real coefficients but $a^2 + b^2 = 1$ so that the state is correctly normalized. Take A_1, A_2 to be a linear combination of σ_z and σ_x acting on the first qubit and take B_1, B_2 to be a linear combination of σ_z and σ_x acting on the second qubit so that all four operators have operator-norm 1. Find the largest possible value of

$$\langle C_{AB}\rangle = \langle\Psi|A_1(B_1 + B_2) + A_2(B_1 - B_2)|\Psi\rangle.$$

Check that your answer reduces to the expected limits when $a = 1, b = 0$ and when $a = b = \frac{1}{\sqrt{2}}$.
[Hint: This is a hard problem if you attempt it using brute force so you may want to consult quant-ph/0611001.]