The Black Hole Information Paradox
Assignment 7

Due Saturday, 27 April 2021

1. **Adjoints of mirrors and mirrors of adjoints**
   Consider a typical equilibrium state, $|\Psi\rangle$. Recall that the mirror of an operator A_i in the little Hilbert space H_Ψ was defined to satisfy
   $$\widetilde{A_i} A_q|\Psi\rangle = A_q e^{-\frac{\beta H}{2}} A_i^\dagger e^{\frac{\beta H}{2}}|\Psi\rangle. \tag{1}$$
   for any simple operator, A_q. Recall that typical states are time-independent and also satisfy
   $$\langle\Psi|A_q|\Psi\rangle = \text{Tr}(e^{-\beta H} A_q), \tag{2}$$
   to a good approximation.

   Use these facts to show that the adjoint of a mirror is the same as the mirror of the adjoint. In other words show that
   $$\langle\Psi|A_i(\widetilde{A_j}^\dagger)A_q|\Psi\rangle = \langle\Psi|A_i(\widetilde{A_j^\dagger})A_q|\Psi\rangle, \tag{3}$$
   where A_i, A_j, A_q are all simple operators.

2. **Commutants using antilinear maps**
   On the little Hilbert space, define
   $$SA_i|\Psi\rangle = A_i^\dagger|\Psi\rangle. \tag{4}$$
   Note that S is an antilinear operator, whose action on the little Hilbert space is completely specified by the equation above. Show that
   $$[SA_iS, A_j]A_q|\Psi\rangle = 0, \tag{5}$$
   for any simple operators A_i, A_j, A_q.

3. **The Modular operator**
   For an antilinear map, the Hermitian conjugate is defined via
   $$\big(|A\rangle, S^\dagger|B\rangle\big) = \left(|B\rangle, S|A\rangle\right), \tag{6}$$
   where $(,)$ is the inner-product between vectors. Note the additional complex conjugation that appears above compared to the definition of the Hermitian conjugate for linear maps. Using this, show that for a typical equilibrium state, the *modular operator* defined via
   $$\Delta = S^\dagger S, \tag{7}$$
   is well approximated by
   $$\Delta = e^{-\beta H}. \tag{8}$$
   Hint: Use a basis of simple operators that comprises polynomials in the creation and annihilation operators.

4. **Tomita-Takesaki definition of Mirrors**
   Show that the operator J defined via

$$S = J\Delta^{\frac{1}{2}}, \tag{9}$$

   satisfies

$$J\Delta^{\frac{1}{2}} = \Delta^{-\frac{1}{2}}J, \tag{10}$$

   and

$$J^2 = 1. \tag{11}$$

   Check that the definition of the mirror operators in an equilibrium state is equivalent to

$$\widetilde{A} = JAJ. \tag{12}$$

5. **Mirror operators in a spin chain**
   Consider a "spin chain" with $N = 5$ sites, where each site hosts a two-state system. At each site we have the three Pauli spin operators σ_a^i where $i = 1 \ldots N$ and $a = x, y, z$. We define "simple" operators as polynomials of order at most 1 in these elementary spin operators. This space has dimension $3N + 1$. More explicitly, a basis of simple operators is given by the identity operator and the set of single Pauli operators at each site.

   Write a small computer script to generate a random state, $|\Psi\rangle$, in the 2^N dimensional Hilbert space. Generate the little Hilbert space about $|\Psi\rangle$ and find the mirror operators $\widetilde{\sigma}_a^i$. Check that although

$$[\widetilde{\sigma}_a^i, \sigma_b^j] \neq 0, \tag{13}$$

   as an operator, we still have

$$[\widetilde{\sigma}_a^i, \sigma_b^j]|\Psi\rangle = 0. \tag{14}$$