7 April 2020

Lecture 23: Interior reconstruction in AdS/CFT

In the last lecture, we described paradoxes that suggested that typical microstates of large AdS black holes did not have smooth interiors

We will turn shortly to a resolution of these paradoxes via state dependence

However, to start with, we do expect that, despite the paradoxes, at least some microstates do have smooth interiors.

Say $|\Psi\rangle$ is such a microstate.

How do we describe the bh interior?

The question is as follows.

In AdS/CFT, we have a well-defined boundary theory and a well-defined space of operators.

So if we consider a bulk scalar field

say $\phi(t, r, \Omega)$

it must map to some operators in the CFT.

What are those operators, when $r < r_{horizon}$?

First, we remind ourselves of the mapping when $r > r_{hor}$.

Consider a bulk scalar field, $\phi(t, r, \Omega)$

There is a boundary operator dual to this field

$$O(t, \Omega) = \lim_{r \to \infty} r^{\Delta} \phi(t, r, \Omega)$$

Consider the Fourier modes of this operator

$$O_{\omega, l}$$

Frequency

Spherical harmonic.

Then outside the horizon, we simply write

$$\phi(t, r, \Omega) = \sum O_{\omega,l} \, e^{-i\omega t} \, Y_l(\Omega) \, f_{\omega,l}(r)$$

where

$f_{\omega,l}$ is fixed by $(\Box - m^2)\phi = 0$

and $\lim_{r \to \infty} f_{\omega,l}(r) = \frac{1}{r^D}$

a) We will now <u>drop</u> the "l" index, since it plays no role in our setting

b) As usual, we can define $a_\omega = O_\omega / \sqrt{G_\omega}$ so that $[a_\omega, a_\omega^\dagger] = 1$. These are what enter the expression for ϕ.

c) There are some subtleties in how to interpret this in position space, but otherwise the mapping above completely solves the problem outside the horizon.

So the question now is:

What is the description of the $\tilde{\alpha}_\omega$ in the boundary theory?

Obviously, we cannot set $\tilde{\alpha}_\omega \propto O_\omega$ and so we need some other idea.

The idea is to define operators that have the "right correlators" with aw

To make this more precise, we need the notion of

a) the little Hilbert space

b) Equilibrium states.

Little Hilbert Space : Background

The key physical point is that the operators $\tilde{a}_\omega$ need to have the right properties only within

simple correlation functions.

This means that we expect that

$$\langle\Psi| A \ \phi(r_{\text{inside}}) B |\Psi\rangle$$

to take on certain values provided A and B are not too complicated.

For instance, if we choose $B|\Psi\rangle = |\Omega\rangle$

AdS vacuum

then we have <u>no expectations</u>
for what the correlator of $\phi(r_{inside})$ should be.

What do we mean by a simple operator?

We are looking for a physical notion of

"simple" $\longrightarrow$ corresponding to physically feasible experiments for an infalling observer.

But if A_{sim} is the set of simple operators, we also look for some mathematical properties

1) $\mathcal{A}_{sim}$ is a complex vector space of operators closed under Hermitian conjugation.

$$A_i \in \mathcal{A}_{sim} \Rightarrow A_i^{\dagger} \in \mathcal{A}_{sim}$$

$$A_i, A_j \in \mathcal{A}_{sim} \Rightarrow c_1 A_i + c_2 A_j \in \mathcal{A}_{sim}.$$

2) We require that

$$\dim(\mathcal{A}_{sim}) \ll e^S$$

where e^S is the entropy of the bh we are interested in

3) For now, we exclude the Hamiltonian and other conserved charges.

$$H \notin \mathcal{A}_{sim}.$$

This is not because these are not simple observables

It is only because they require special treatment

4) A_{sim} is not an algebra.

$$A_i, A_j \in A_{sim} \not\Rightarrow A_i A_j \in A_{sim}$$

But we would like that for "many" observables, products are also within the set.

There is no unique choice of A_{sim}.

In arXiv: 1310.6335 where these issues were first discussed, the following choice of A_{sim} was made.

Let a_w be the operators described earlier [modes of boundary operators, with rescaling]

Consider polynomials in these operators with some <u>cutoff on their order</u>

$$A_{sim} = \mathrm{span}\{a^\dagger_{w_1}, a_{w_1}, a_{w_1} a_{w_2}, a^\dagger_{w_1} a_{w_2}, \ldots \\ a_{w_1} a_{w_2} \cdots a_{w_q} a^\dagger_{w_{q+1}} \cdots a^\dagger_{w_{q+p}}, \ldots\}$$

the <u>cutoff</u> is necessary to ensure

$$\dim(A_{sim}) \ll e^S$$

We see this set also has the other desired properties

For low order polynomials, A_i, A_j

$$A_i A_j \in A_{sim}$$

but if A_i, A_j have high enough order

$$A_i A_j \notin A_{sim}$$

It is possible to make other choices for A_{sim} (i.e. not choose the set of low-order polynomials or instead consider

projection operators or bounded operators or something else ...)

Here we will continue to think in terms of polynomials.

<u>No</u> <u>annihilation</u> <u>property</u>

Since $\dim(A_{sim}) \ll e^S$, we expect that

$$A_i |\psi\rangle \neq 0 \, , \quad \forall A_i \in A_{sim}$$

There is a simple physical reason for this: we don't expect simple operations by an infalling observer to annihilate the state.

More mathematically in a e^S dimensional space, we expect the null-set of a set of operators of dim $\ll e^S$ to have <u>high codimension</u>.

So only a set of measure 0 is expected to be annihilated by the A_i.

Nevertheless, there are some states that may be annihilated by elements of A_{sim}.

$A_i|\psi\rangle = 0$ for $A_i \in A_{sim}$ $\Rightarrow$ we cannot construct the interior and we do not expect a smooth horizon.

Little Hilbert Space

We can now finally define the little Hilbert space

$$\mathcal{H}_\psi = \mathcal{A}_{sim} |\psi\rangle$$

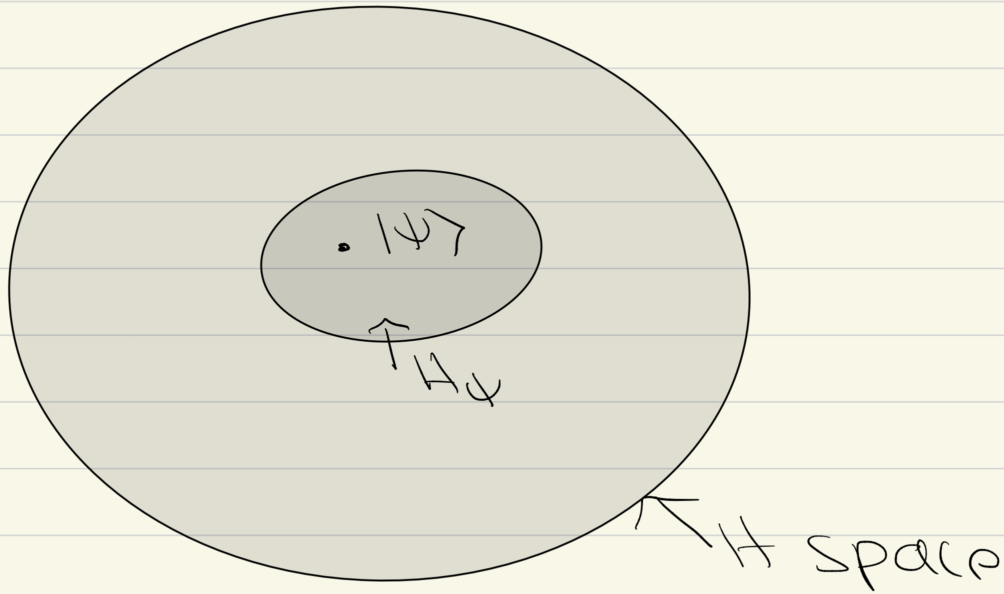
i.e. the space of states obtained by acting on the original microstate with all possible simple operators.

This is also called the code subspace but this concept was originally introduced in the context of black holes and later adopted to quantum error correction.

Notice

$$\dim(H_\psi) = \dim(A_{sim}) = D \ll e^S$$

We can picture this as follows



We need one more notion before we define interior operators:

<u>equilibrium states</u>

Physically we want to consider states where the black hole has settled down and distinguish them from those where it has not.

One way to define such states is to consider

$$\chi_p(t) = \langle \psi | e^{iHt} A_p e^{-iHt} | \psi \rangle$$

$A_p \in A_{sim}$

Then, we would like

$$|X_p(t) - X_p(0)| \sim O(e^{-S/2})$$

for long-enough times t.

[Recall this is what we expect for observables long enough after the perturbation. See last lecture]

So equilibrium states are those that are time-independent. For this discussion

This notion may require refinement but this is an open problem

Mirror operators

We are now in a position to define the interior operators for an equilibrium state.

Pick a basis of operators for $\mathcal{H}_\psi$.

Let denote elements of this basis by

$$|v_m\rangle = A_m |\psi\rangle, \quad A_m \in \mathcal{A}_{sim}$$

Then we define the operator

$$\tilde{a}_\omega |v_m\rangle = |u_m\rangle$$

where $|u_m\rangle = A_m e^{-\beta\omega/2} a^\dagger_\omega |\psi\rangle$

This definition should be extended to all of H_V by <u>linearity</u>

More specifically set g^{mn} so that

$$\sum_{n=1}^{D} g^{mn} \langle v_n | v_p \rangle = \delta^m_p$$

Then

$$\tilde{a}_W = \sum_{m,n=1}^{D} g^{mn} |u_m\rangle\langle v_n|$$

Similarly

$$\tilde{a}^{+}_{\omega} \, |V_m\rangle = e^{\beta\omega/2} A_m \, a_{\omega,e} |\psi\rangle$$

This definition automatically specifies how a polynomial of the mirror operators acts

For instance,

$$\tilde{a}_\omega \tilde{a}^{+}_{\omega} \, |\psi\rangle = \tilde{a}_\omega \, e^{\beta\omega/2} a_\omega |\psi\rangle.$$

[Using rule for $\tilde{a}^{+}_{\omega}$]

Now we use the rule for $\tilde{a}_\omega$

$$\tilde{a}_\omega e^{\beta\omega/2} a_\omega |\psi\rangle = e^{\beta\omega/2} a_\omega e^{-\beta\omega/2} a^{+}_{\omega} |\psi\rangle$$

$$= a_\omega a_\omega^\dagger |\psi\rangle$$

In general we can also use

$$\tilde{A}_n A_m |\psi\rangle = A_m e^{-\beta H/2} A_n^\dagger e^{\beta H/2} |\psi\rangle.$$

Two-point functions

Our definition of mirror operators satisfies the correlators required for a smooth horizon

Notice

$$\langle\psi|\, \tilde{a}_\omega \tilde{a}_\omega^\dagger \,|\psi\rangle = \langle\psi|\, a_\omega\, a_\omega^\dagger \,|\psi\rangle$$

[By the previous calculation]

$$= \frac{1}{1-e^{-\beta\omega}}$$

Also not

$$\langle\psi|\, \tilde{a}_\omega\, a_\omega \,|\psi\rangle = \langle\psi|\, a_\omega\, a_\omega^\dagger \,|\psi\rangle e^{-\beta\omega/2}.$$

$$= \frac{e^{-\beta\omega/2}}{1-e^{-\beta\omega}}$$

as required.

Commutators

An important aspect of this "mirror-operator" construction was that it ensured that the mirror operator commuted with simple operators inside correlators

Historically, the puzzle was as follows [which AMPSS articulated.]

Consider a quantum field with operators a_{ω_i} and $a^\dagger_{\omega_i}$ [discretized]

Then there is no operator

that commutes exactly with a_{ω_i} and $a^\dagger_{\omega_i}$ since polynomials of these operators generate the entire algebra.

However notice that

$$\tilde{a}_\omega A_n |\psi\rangle = e^{-\beta\omega/2} A_n a_\omega^\dagger |\psi\rangle$$

$$= A_n \tilde{a}_\omega |\psi\rangle$$

so

$$[\tilde{a}_\omega, A_n] |\psi\rangle = 0$$

Therefore the commutator of the mirror operators with simple operator annihilates the state.

Therefore within simple correlators

$$\langle \psi | A_m [\tilde{a}_\omega , A_q] A_p | \psi \rangle = 0$$

But

$$[\tilde{a}_\omega , A_q] \neq 0$$ as an operator.