4 Feb 2021

Lecture 8: Some Quantum Information Results

These arguments led Mathur to propose a more formidable paradox suggesting that such "small corrections" cannot resolve the information paradox.

Later, AMPS elaborated this paradox slightly

The main point of the argument was as follows

1) A smooth horizon requires mode requires entanglement between modes outside and inside the horizon

2) Typicality requires entanglement between modes outside and modes far away.

3) Entanglement is monogamous. So, this is a paradox.

The important difference is that this does not involve only the exterior but also makes refence to the $\tilde{A}_{\omega}$ operators

We will take a detour into quantum information to review this paradox

1) We will quantify entanglement in terms of Bell correlators [explain adv. over E.E.]

2) explore average entanglement

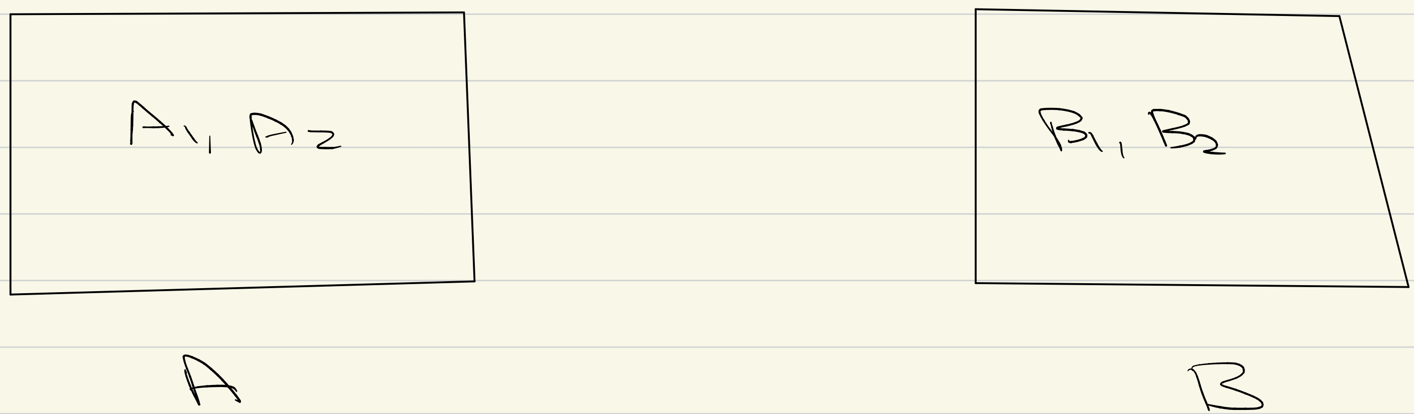
CHSH correlators Tsirelson's bound.

The key aspect of entanglement is that quantum mechanically some correlators can exceed possible classical correlators.

So entangled systems can violate Bell's inequalities.

It is more convenient to frame this in terms of CHSH correlators.

Say we have two systems — A and B



In each system we have two observables A_1, A_2 in A

and

B_1, B_2 in B.

Imp restriction.

Each observable takes values in $\{-1, 1\}$

For instance A and B could be two coins

Upon opening the box and observing we can assign

$$A_1 = \begin{cases} 1 & \text{if heads} \\ -1 & \text{if tails.} \end{cases} \qquad A_2 = \begin{cases} 1, & \text{if Rs 5 coin} \\ -1, & \text{if Rs 10 coin.} \end{cases}$$

and similarly for B_1 and B_2

Now define the joint observable

$$C_{AB} = A_1(B_1 + B_2) + A_2(B_1 - B_2)$$

First consider the classical case.

Classical means we can assign values simultaneously to all observables

Then it is easy to check that

$$|C_{AB}| \leq 2.$$

because

$$|C_{AB}| \leq |B_1+B_2| + |B_1-B_2|$$

(since $|A_1|, |A_2| \leq 1$)

$$\leq 2 \max(|B_1|, |B_2|)$$

$$\leq 2.$$

[Give numerical example]

Tsirelson showed that, quantum-mechanically,

$$|C_{AB}| \leq 2\sqrt{2}$$

If

$$2 \leq |C_{AB}| \leq 2\sqrt{2}$$

we can say the systems are entangled.

Here is one way to visualize how this can happen.

Say we have a state $|\psi\rangle$.

Let

$$|A_1\rangle = A_1|\psi\rangle; \quad |A_2\rangle = A_2|\psi\rangle, \quad |B_1\rangle = B_1|\psi\rangle$$

$$|C_1\rangle = C_1|\psi\rangle$$

Note

$$|B_1\rangle + |B_2\rangle = \sqrt{2}\,|A_1\rangle$$

$$|B_1\rangle - |B_2\rangle = \sqrt{2}\,|A_2\rangle$$

but $\quad \langle A_i|A_i\rangle = \langle B_i|B_i\rangle = \underline{1}.$

For this to happen it is crucial that

$$[B_1, B_2] \neq 0$$

$$[A_1, A_2] \neq 0$$

we cannot assign definite values to A_1, A_2 simultaneously

But, of course

$$[A_i, B_j] = 0$$

(Def^n of distinct systems)

In fact, say $A_1^2 = A_2^2 = B_1^2 = B_2^2 = 1$ then

$$C_{AB}^2 = A_1^2 (B_1+B_2)^2 + A_2^2 (B_1-B_2)^2 + A_1 A_2 (B_1+B_2)(B_1-B_2) + A_2 A_1 (B_1-B_2)(B_1+B_2) = 4 - [A_1, A_2][B_1, B_2]$$

Monogamy of entanglement

Now say we have a third system, C, which has its own observables C_1, C_2

By C being a distinct system we again mean

$$\{C_i, A_j\} = \{C_i, B_j\} = 0$$

then we can define

$$C_{AC} = A_1(C_1 + C_2) + A_2(C_1 - C_2)$$

which can be used to measure entanglement between A and C.

Then we have a remarkable inequality

$$\langle C_{AB} \rangle^2 + \langle C_{AC} \rangle^2 \leq 8$$

Cases:

$$|C_{AB}| > 2 \Rightarrow |C_{AC}| < 2$$

$$|C_{AC}| > 2 \Rightarrow |C_{AB}| < 2$$

$$C_{AB} = 2\sqrt{2} \Rightarrow C_{AC} = 0$$

max ent between A&B $\Rightarrow$ no ent between A & C!

This quantifies the monogamy of entanglement!

We now turn to the average entanglement between subsystems

The question we want to ask is the following, Say we have a big system, made up of two parts

one part with a H-space of dim $e^{S'}$

another part with H-space of dim e^{S}

We will assume for simplicity that

$$e^{S'} \ll e^{S}$$

For large systems, this is not a strong assumption

eg. say one system has 10^{10} qubits

the other has $(1 - 10^{-6}) \times 10^{10}$ qubits

$$e^S = 2^{10^{10}}$$

$$e^{S'} = 2^{-10^4} \times 2^{10^{10}}$$

so $e^{S'} << e^S$

We want to ask about the entanglement between the two

a) Given operators satisfying some simple properties in the smaller subsystem, we can find ops in the larger subsystem satisfying Tsirelson's bound.

b) Entanglement entropy between the two subsystems

Lets start by examining the density matrix.

Consider a "typical" state in the larger system

$$|\psi\rangle = \sum_{m=1}^{e^{s'}} \sum_{n=1}^{e^{s}} a_{mn} |m,n\rangle .$$

the smaller density matrix is

$$\rho_{mm'} = \sum_{n=1}^{e^{s}} a_{mn} a^{*}_{m'n}$$

The larger density matrix is

$$\rho_{nn'} = \sum_{m=1}^{e^{S'}} a_{mn} a^{*}_{mn'}$$

But the eigenvalues of the larger density matrix are **the same** as the eigenvalues of the smaller one.

This is because we can always use singular value decomposition for the matrix a_{mn} [perform a change of basis in both the smaller and larger system] to write the state as

$$|\psi\rangle = \sum_{\alpha=1}^{e^{S'}} \sqrt{P_\alpha}\, |\alpha, \alpha\rangle.$$

so the larger density matrix has rank at most $e^{S'}$

the "expectation value" of $P_{mm'}$ is

$$\int \sum_n \int a_{mn} a^*_{m'n} \, d\mu_\psi = \sum_n \frac{1}{e^{S+S'}} \delta_{mm'} \delta_{nn'}$$

$$= \frac{1}{e^{S'}}$$

But notice that

$$\delta = \frac{1}{e^{S'}} \left\langle \sum_{mm'} \left(P_{mm'} - \frac{1}{e^{S'}} \delta_{mm'} \right)^2 \right\rangle$$ gives the

average deviation of each eigenvalue.

Now

$$\left\langle \sum_{m,m'} \left| P_{mm'} - \frac{1}{e^{s'}} \delta_{mm'} \right|^2 \right\rangle$$

$$= \left\langle \sum_{m,m'} P_{mm'} P_{m'm} - \frac{2}{e^{s'}} \sum_{m} P_{mm} + \frac{1}{e^{s'}} \right\rangle .$$

$$\left\langle \sum_{nq,mm'} a_{mn} a^*_{m'n} a_{m'q} a^*_{mq} \right\rangle - \frac{1}{e^{s'}}$$

Recall

$$\langle a_i a_j^* a_k a_l^* \rangle = \frac{1}{w(w+1)} \left[\delta_{ij} \delta_{kl} + \delta_{il} \delta_{jk} \right]$$

so we get above.

$$e^{s+s'} \frac{1}{(e^{s+s'}+1)} \left(\delta_{nn} \delta_{qq} \delta_{mm'} + \delta_{mm} \delta_{m'm'} \delta_{nq} \right) - \frac{1}{e^{s'}}$$

$$e^{s+s'} \frac{1}{(e^{s+s'}+1)} \left(\delta_{nn}\, \delta_{qq}\, \delta_{mm'} + \delta_{mm}\, \delta_{m'm'}\, \delta_{nq}\right) - \frac{1}{e^{s'}}$$

$$= \frac{e^{s}+e^{s'}}{e^{s+s'}+1} - \frac{1}{e^{s'}}$$

Recall the additional normalization in δ

$$\delta = \frac{1}{e^{s'}}\left[\frac{e^{s}+e^{s'}}{e^{s+s'}+1} - \frac{1}{e^{s'}}\right]$$

At large s, s'

$$\delta \approx \frac{1}{e^{s+s'}}$$

$\sqrt{S}$ gives us the average deviation of the eigenvalues of ρ from $e^{-S'}$. For $e^{S} >> e^{S'}$ we see that this deviation is much smaller than the size of the eigenvalue.

In this sense, the density matrix is close to the identity matrix.

Note that this argument does not work for the larger system. There the deviations are larger than the size of eigenvalues.

To be expected since the larger density matrix can have rank at most $e^{S'}$.

Now say that we have two "pseudospin" operators, A_1, A_2, on the first system

This means we have operators that share the following properties of σ_3 and σ_x

$$A_1^2 = A_2^2 = 1$$

$$(A_1 + A_2)^2 = (A_1 - A_2)^2 = 2$$

One can always find such operators by considering a 2d subsector of the Hilbert space.

We can find such operators in a SHO also as we will see.

Now, as mentioned above, we can choose a Schmidt basis so that the state looks like

$$|\psi\rangle = \frac{1}{e^{s'/2}} \sum_{m=1}^{e^{s'}} |m, \tilde{m}\rangle.$$

This uses our result

Let the matrix elements of A_1, A_2 be

$$A_1|m\rangle = \sum_q (A_1)_{mq} |q\rangle \; ; \quad A_2|m\rangle = \sum_q (A_2)_{mq} |q\rangle.$$

Now **define**

$$\tilde{A}_1|\tilde{m}\rangle = \sum_q (A_1)_{qm} |\tilde{q}\rangle \; ; \quad \tilde{A}_2|\tilde{m}\rangle = \sum_q (A_2)_{qm} |\tilde{q}\rangle$$

Since these are just the transpose of the A_1, A_2 matrices, we also have

$$\|\tilde{A}_1\| = \|\tilde{A}_2\| = 1$$

But also

$$\tilde{A}_1 |\psi\rangle = \frac{1}{e^{s'/2}} \sum_{m=1}^{e^{s'}} \tilde{A}_1 |m, \tilde{m}\rangle$$

$$= \frac{1}{e^{s'/2}} \sum_{m,q} (A_1)_{qm} |m, \tilde{q}\rangle$$

rename $m \leftrightarrow q$

$$\tilde{A}_1 |\psi\rangle = \frac{1}{e^{s'/2}} \sum_{m,q} (A_1)_{mq} |q, \tilde{m}\rangle$$

$$= A_1 |\psi\rangle$$

Similarly

$$\tilde{A}_2 |\psi\rangle = A_2 |\psi\rangle$$

Set

$$B_1 = \frac{1}{\sqrt{2}}(\tilde{A}_1 + \tilde{A}_2) \; ; \quad B_2 = \frac{1}{\sqrt{2}}(\tilde{A}_1 - \tilde{A}_2)$$

Clearly $B_1^2 = B_2^2 = \underline{1}$

Now define

$$C_{AB} = A_1(B_1 + B_2) + A_2(B_1 - B_2)$$

Clearly

$$\langle\psi|C_{AB}|\psi\rangle = \sqrt{2}\left(\langle\psi|A_1^2|\psi\rangle + \langle\psi|A_2^2|\psi\rangle\right)$$

$$= 2\sqrt{2}.$$

Next, consider the entanglement entropy

For the identity density matrix, we clearly have

$$-\mathrm{tr}(\rho \ln \rho) \approx S'$$

In general, we can write

$$-\mathrm{tr}(\rho \ln \rho) \approx \min(S', S)$$

as long as

$$|S' - S| \gg 1.$$

We can plot this as a ratio of the fraction of the size of one system to the entire system