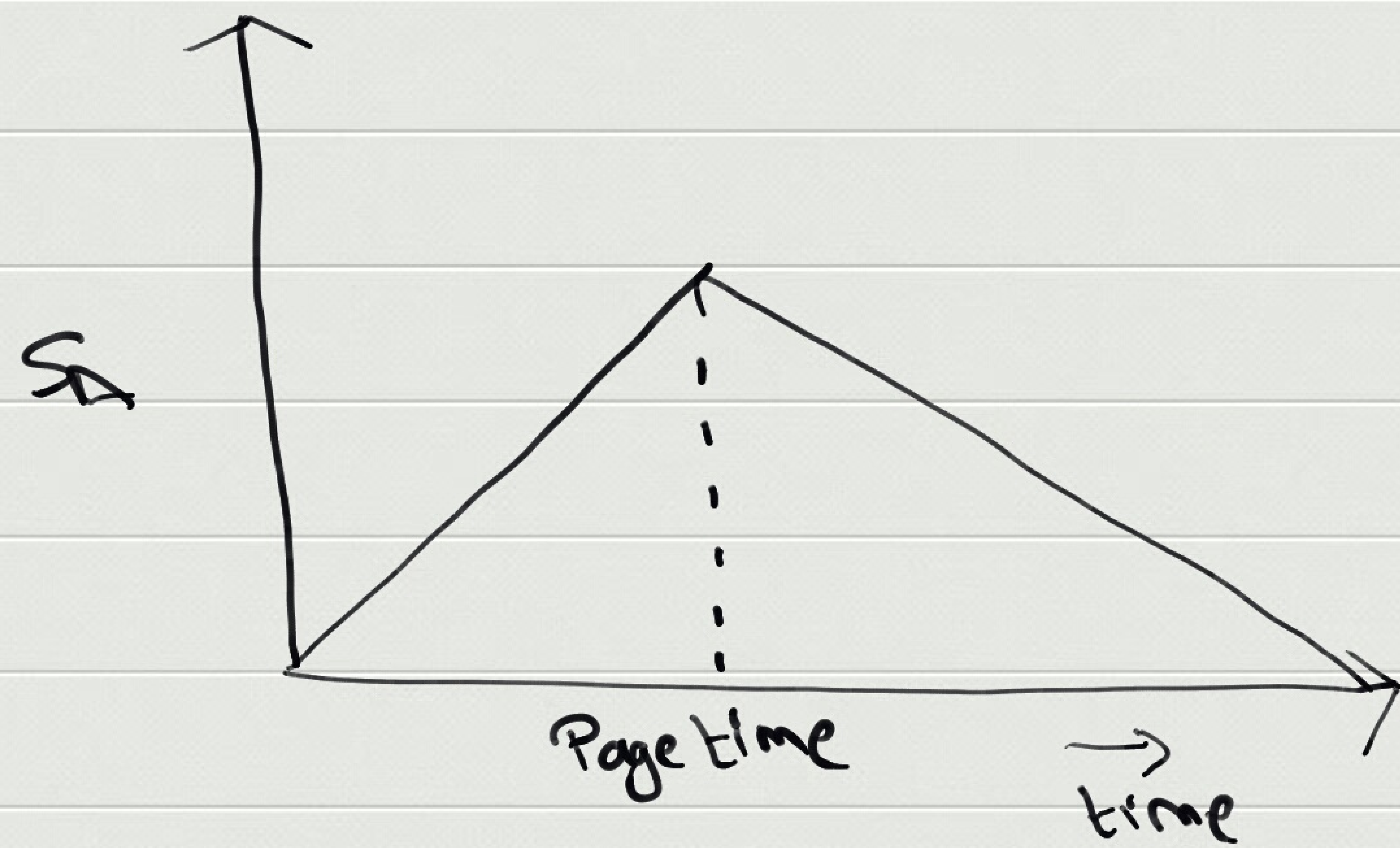
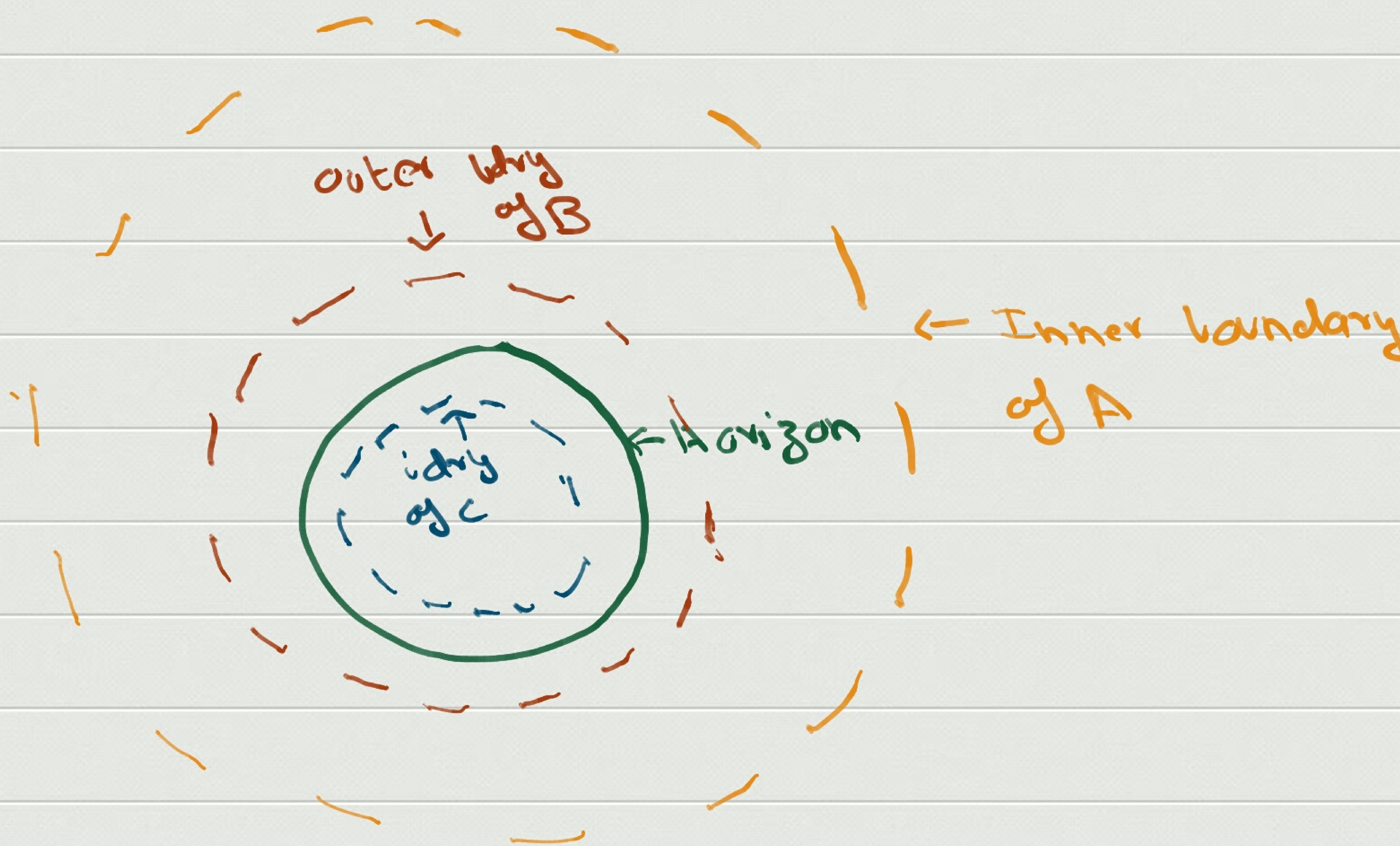
Yesterday we ended on the Page curve.



Mathur (2009) pointed out that this leads to a paradox.

Let us create 3 imaginary regions



They are

A - very far away region

B - near horizon region outside the B.H

C - near horizon region inside the B.H

Now consider a time beyond the Page time. Then we have

$$S_{AB} = -tr(\rho_{A\cup B} \ln \rho_{A\cup B}) < S_A = -tr(\rho_A \ln \rho_A)$$

because $S_{AB}(t) \sim S_A(t+\delta t)$

So

$S_{AB} < S_A$: UNITARITY & GENERICITY

Next we also have that

$S_B > 0$; $S_c > 0$: DENSITY MATRIX OF BOTH SIDES OF THE HORIZON IS MIXED

But also that

$S_{Bc} < S_c$; $S_{Bc} < S_B$: MODES ACROSS THE HORIZON PURIFY EACH OTHER

These 3 robust conclusions turn out to be in contradiction with the strong subadditivity of entropy

This states that for any 3 separate systems A, B, C we have

$$S_{AB} + S_{BC} \geq S_A + S_C$$

Here this is clearly violated at $O(1)$. We have

$$S_{AB} < S_A \quad \text{by } O(1)$$

and

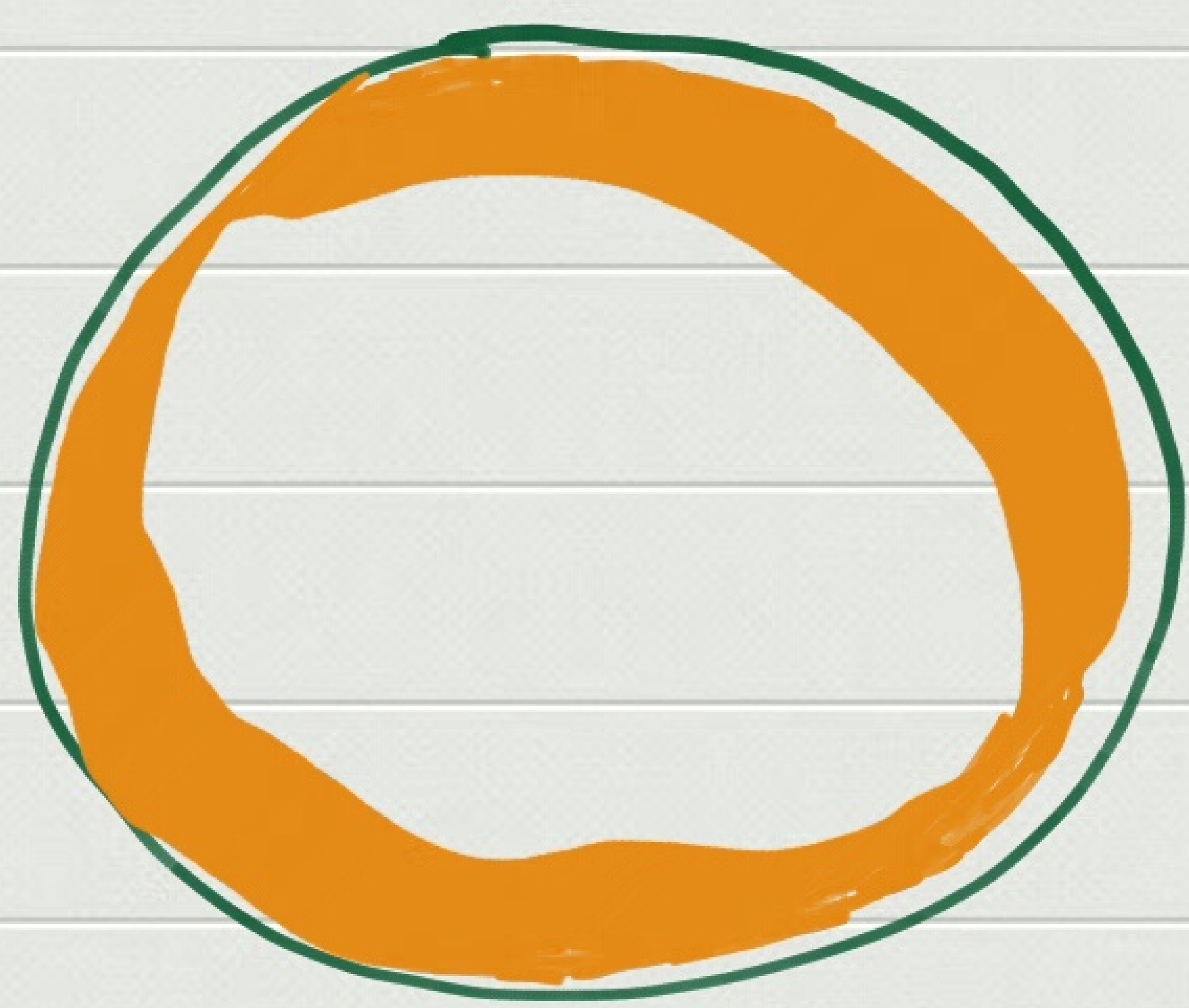
$$S_{BC} < S_C \quad \text{by } O(1).$$

The people who framed the paradox suggested that we drop

$$S_{BC} < S_C$$

But if we drop this, we lose the conclusion that the horizon is smooth. Instead we get a firewall!

or a fuzzball!



Both these conclusions violate effective field theory and should not be accepted unless we have no other option!

Two more precise arguments for a paradox.

1) Consider the $\tilde{b}_\omega$ modes inside the horizon. Recall that the mode they multiply is

$b_\omega e^{-i\omega(r_* - t)}$

So they have negative energy!

More precisely

$$[H, \tilde{b}_\omega] = \omega \tilde{b}_\omega$$

$$[H, \tilde{b}_\omega^\dagger] = -\omega \tilde{b}_\omega^\dagger$$

Now take some generic B.H state

We would like

$$\langle \tilde{b}_w \tilde{b}_w^+ \rangle = \frac{1}{1-e^{-\beta w}}$$

But we also expect

$$\langle \tilde{b}_w \tilde{b}_w^+ \rangle = \frac{\mathrm{Tr}}{Z}\left(e^{-\beta H} \tilde{b}_w \tilde{b}_w^+\right) = \frac{\mathrm{Tr}}{Z}\left(\tilde{b}_w^+ \, e^{-\beta H} \, \tilde{b}_w\right).$$

$$= \frac{\mathrm{Tr}}{Z}\left(e^{-\beta H} \tilde{b}_w^+ \tilde{b}_w\right) e^{-\beta w} = \frac{\mathrm{Tr}}{Z}\left(e^{-\beta H}\left(\tilde{b}_w \tilde{b}_w^+ - 1\right)\right) e^{-\beta w}$$

So $$\langle \tilde{b}_w \tilde{b}_w^+ \rangle \overset{?}{=} \frac{-e^{-\beta w}}{1-e^{-\beta w}}$$ ← Absurd!

Also consider the Unruh number operator. With

$$\alpha_w = \tilde{b}_w - e^{-\beta w/2} b_w^{+}$$
$$\gamma_w = b_w - e^{-\beta w/2} \tilde{b}_w^{+}$$

This is

$$N_a = \alpha_w^{+} \alpha_w + \gamma_w^{+} \gamma_w$$

We expect that $N_a = 0$. On the other hand, take

$$N_u = b_w^{+} b_w$$

We expect

$$\langle N_u \rangle = \frac{e^{-\beta w}}{1-e^{-\beta w}}$$

But in N_ν eigenstates, we expect $\langle N_a \rangle = Or(1)$.
Now we expect

$$\mathrm{Tr}_{mrc}(N_a) = 0$$

in the microcanonical ensemble.
Since $[N_\nu, H] \approx 0$, we might want to change bases and evaluate the trace in the number eigenstate basis. But since N_a is a positive operator

$$\sum_{N_\nu} \langle N_\nu | N_a | N_\nu \rangle = or(1)!$$

Framers of the paradox would like to claim that

$\tilde{\mathcal{O}}_\omega$ do not exist!

Solution:

We should demand the correct properties of $\tilde{\mathcal{O}}$ only in low-pt correlators about a given state!

Let us make this precise. Consider a given B.H. state $|\Psi\rangle$. and an infalling observer. He can do a limited number of experiments. Can measure

$$\langle \Psi | A_\alpha | \Psi \rangle .$$

where

A_α is a finite polynomial in the modes $\tilde{\mathcal{O}}_\omega$.

eg. observer can measure

$$\langle \Psi | b_\omega^\dagger b_\omega | \Psi \rangle$$

or

$$\langle \Psi | b_\omega^\dagger b_\omega b_{\omega'}^\dagger b_{\omega'} | \Psi \rangle$$

But, <u>not</u>

$$\langle \Psi | b_{\omega_1} \cdots\cdots\cdots b_{\omega_S} | \Psi \rangle$$

where S is the B.H. entropy and cannot measure modes with

$$\omega \sim M_{Pl}$$

Lets call $\mathcal{A}$ the full set of observables a reasonable observer can measure.

$A_\alpha \in \mathcal{A}$

Then, we define

$$\tilde{\iota}_w A_\alpha |\psi\rangle = e^{-\beta w/2} A_\alpha \iota_w^+ |\psi\rangle.$$

$$\tilde{\iota}_w^+ A_\alpha |\psi\rangle = e^{\beta w/2} A_\alpha \iota_w |\psi\rangle$$

If we impose this $\forall A_\alpha$ it gives $\dim(\mathcal{A})$ linear equations for $\tilde{\iota}_w$

But remember $\tilde{\iota}_w$ operates in a $e^S \times e^S$ dim space.
So provided

$$\dim(\mathcal{A}) < e^S$$

relevance of restricting observables!

We can solve these equations! This trivial definition

RESOLVES ALL PARADOXES!

Check

Consider

$$\tilde{b}_\omega \tilde{b}_\omega^{+} |\psi\rangle = \tilde{b}_\omega b_\omega |\psi\rangle e^{\beta\omega/2} \leftarrow \text{Using rule for } \tilde{b}_\omega^{+}$$

$$= b_\omega b_\omega^{+} |\psi\rangle$$

$\uparrow$ Using rule for $\tilde{b}_\omega$

so

$$\langle\psi| \tilde{b}_\omega \tilde{b}_\omega^{+} |\psi\rangle = \langle\psi| b_\omega b_\omega^{+} |\psi\rangle = \frac{1}{1-e^{-\beta\omega}}$$

Also

$$\tilde{b}_\omega b_\omega |\psi\rangle = e^{-\beta\omega/2} b_\omega b_\omega^\dagger |\psi\rangle$$

so

$$\langle\psi| \tilde{b}_\omega b_\omega |\psi\rangle = \frac{e^{-\beta\omega/2}}{1-e^{-\beta\omega}}$$

How did we evade the impossibility theorem?

$$\langle\psi| \tilde{b}_\omega \tilde{b}_\omega^\dagger |\psi\rangle \neq \mathrm{Tr}\left(e^{-\beta H} \tilde{b}_\omega \tilde{b}_\omega^\dagger\right)$$

because $\tilde{b}_\omega$ depends on the state.

So given a basis of states

$$\sum_i \langle \psi_i | \tilde{\imath}_w \tilde{\imath}_w^{\dagger} | \psi_i \rangle \neq \mathrm{tr}(\tilde{\imath}_w \tilde{\imath}_w^{\dagger})$$

because $\tilde{\imath}_w$ secretly depends on the state.

The correct local operators to use depend on the state of the system.

Strong subadditivity paradox is resolved by the observation that

A, B, C are not independent in quantum gravity!

The degrees of freedom inside the black hole are scrambled versions of dof outside!

We can make this precise using correlators

$$[b_w, \tilde{b}_w]\,|\psi\rangle = b_w \tilde{b}_w\,|\psi\rangle - \tilde{b}_w b_w\,|\psi\rangle$$

$$= b_w \tilde{b}_w\,|\psi\rangle - b_w \tilde{b}_w\,|\psi\rangle = 0.$$

Recall the linear eqn defining $\tilde{b}_w$

but the operator

$$C = [b_w, \tilde{b}_w] \neq 0$$

we have

$$C\,|\psi\rangle = 0 \quad \text{and} \quad C A_\alpha\,|\psi\rangle = 0, \ \forall A_\alpha \in \mathcal{A}$$

SLOGAN!

the dof outside and inside are interdependent but in a very clever way. A local observer outside, and a local observer inside think their dof are independent (because the commutator vanishes in low-pt correlators) but if they measure arbitrarily high pt correlators (with more than S-insertions) they will see this loss of locality!