Quantum Aspects of Black Holes, Lecture 12,

28 Sept 2016

Last time we discussed the Penrose process, where we saw that the mass of a Kerr black hole could decrease. We will now analyze the limits of this mass extraction.

It turns out that we cannot reduce the area.

To compute the area, we look at the induced metric on θ, ϕ

$$ds^2 = (r_+^2 + a^2\cos^2\theta)\,d\theta^2 + \left[\frac{(r_+^2+a^2)^2\sin^2\theta}{r_+^2+a^2\cos^2\theta}\right]d\phi^2$$

The area is computed through

$$A = \int\sqrt{\gamma}\,d\theta\,d\phi = 4\pi\,(r_+^2 + a^2).$$

$$r_+^2 + a^2 = 2M^2 + 2\sqrt{M^4 - (J^2 + e^2M^2)} - e^2.$$

Using $r_+ = M + \sqrt{M^2 - \left(\frac{J^2}{M^2} + e^2\right)}$

[Recall that $J = Ma$]

So

$$\frac{\partial A}{\partial M} = \left(4M + \frac{4M^3 - 2e^2 M}{\sqrt{M^4 - (J^2 + e^2 M^2)}} \right) 4\pi$$

$$= 8\pi \frac{(r_+^2 + a^2)}{\sqrt{M^2 - (a^2 + e^2)}}$$

Also define

$$\kappa = \frac{\sqrt{M^2 - (a^2 + e^2)}}{(r_+^2 + a^2)}$$

And note that

$$\frac{\partial A}{\partial J} = \frac{-8\pi J}{\sqrt{M^4 - (J^2 + e^2 M^2)}} = -\frac{8\pi a}{\sqrt{M^2 - (a^2 + e^2)}}$$

Note that

$$\frac{\left(\frac{\partial A}{\partial J}\right)_M}{\left(\frac{\partial A}{\partial M}\right)_J} = \frac{-a}{r_+^2 + a^2} = -\Omega.$$

therefore

$$\delta A = \frac{8\pi}{\kappa} (\delta M - \Omega \delta J)$$

But recall that whatever falls into the black hole has to have locally +ve energy w.r.t. the killing vector $\left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi}\right)$.

But this means

$$\delta M - \Omega \delta J > 0$$

and so $\underline{\delta A > 0}$ in the Penrose process.

This is called the second law of black hole thermodynamics.

It turns out that this is a very general rule valid for all sorts of black holes.

We will now derive the 2^{nd} law. The main input comes from the Raychaudhuri equation, so we will first derive that.

Raychaudhuri Equation

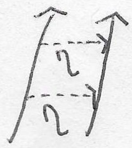
Consider a congruence of null geodesics. This just means a set so that only one geodesic passes through each point.

The tangent vector field is denoted by ξ^a.

We define

$$\nabla_a \xi_b = B_{ab}$$

The significance of B_{ab} is as follows. Consider the deviation vector field from one geodesic in the congruence to a nearby geodesic.



We clearly have

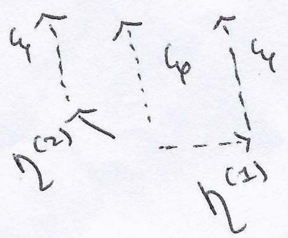
$$L_\xi \eta = 0$$

$$\Rightarrow \quad \xi_a \nabla^a \eta_b - \eta_a \nabla^a \xi_b = 0$$

$$\text{or} \quad \xi_a \nabla^a \eta_b = \eta_a \nabla^a \xi_b = B^a{}_b \eta_a .$$

So the tensor B tells us how the deviation is parallel transported along a geodesic.

Note that the deviation vector field is not unique.



Both $\eta^{(1)}$ and $\eta^{(2)}$ give valid deviations to nearly geodesics in the congruence. But the equation above holds for all of them.

Second note that

$$\xi^a \nabla_a (\xi^b \eta_b) = \underbrace{(\xi^a \nabla_a \xi^b)}_{0} \eta_b + (\xi^a \nabla_a \eta_b)\, \xi^b$$

because ξ is $\parallel$ transported to itself.

$$= \xi^a \nabla_a \eta^b \xi_b - (\eta^a \nabla_a \xi^b)\, \xi_b + (\eta^a \nabla_a \xi^b)\, \xi_b$$

$$= (L_\xi \eta)^b \xi_b + \eta^a \nabla_a \left(\frac{\xi^b \xi_b}{2}\right)$$

$$= \underbrace{0}_{L_\xi \eta = 0} + \underbrace{0}_{\xi^b \xi_b = 0}$$

So the parallel transport of the "non-orth" part of η is uninteresting since it remains constant. Focus on η, with the property that

$$\eta_b \xi^b = 0. \qquad (1)$$

Now when null geodesics are involved we also have

$$\xi_b \xi^b = 0$$

So we consider the equivalence class of deviation vector fields

$$\eta^a \to \eta^a + \lambda \xi^a. \qquad (2)$$

Conditions (1) and (2) mean that we are effectively considering a two dimensional space.

Next, it is useful to consider the induced metric on this 2-d space. ~~given~~

we call this h_{ab}.

You might worry that picking different reps of η will lead to different answers, but

$$g_{ab}\,\eta^a\,(\eta^b + \lambda\,\xi^b) = 0$$

because $(\eta, \xi) = 0$.

Similarly the tensor B also has a well-defined action on this 2-d space because

$$(B^a{}_b\,\eta_a)\,\xi^b = 0.$$

Now we can decompose.

$$B_{ab} = \frac{1}{2}\,\theta\,h_{ab} + \sigma_{ab} + w_{ab}$$

where $\sigma_{ab} = \sigma_{ba}$ and $\sigma_{ab}g^{ab} = 0$ and $w_{ab} = -w_{ba}$.

We now want to analyze $\frac{d\theta}{d\lambda}$.

To do this we write

$$\theta = B_{ab}\,h^{ab}.$$

$$\xi^a\,\nabla_a\theta = \xi^c\,\nabla_c\nabla_a\xi_b\,h^{ab}.$$

$$\xi^c \nabla_c \nabla_a \xi_b = \xi^c \nabla_a \nabla_c \xi_b + R_{cab}{}^d \xi_d \xi^c$$

$$= -(\nabla_a \xi^c)(\nabla_c \xi_b) + (\nabla_a \underbrace{\xi^c \nabla_c \xi_b}_{=0})$$

$$+ R_{cab}{}^d \xi_d \xi^c$$

$$= R_{cab}{}^d \xi_d \xi^c - X_a{}^c X_{cb}.$$

Taking the trace, we get

$$\frac{d\theta}{d\lambda} = \xi^c \nabla_c \theta = -X_a{}^c X_c{}^a - R_{cd} \xi^c \xi^d$$

$$= -\frac{1}{2}\theta^2 - \sigma_{ab}\sigma^{ab} - R_{cd}\xi^c \xi^d + \omega_{ab}\omega^{ab}$$

Now we make some assumptions. First we restrict attention to geodesic congruences that are hypersurface orthogonal. This means that locally we can write

$$\xi_b = f\, dg$$

for some f and g. but then

$$d\xi = df \wedge dg = \frac{df}{f} \wedge \xi.$$

$$\Rightarrow \nabla_a \xi_b - \nabla_b \xi_a = V_a \xi_b - V_b \xi_a$$

for some V.

The restriction of $\nabla_{[a}\xi_{b]}$ to vectors orthogonal to ξ gives ω. but we see here that for n_1, n_2, with $(n_1, \xi) = (n_2, \xi) = 0$

$$\nabla_{[a}\xi_{b]} n_1^a n_2^b = 0$$

so

$$\omega_{ab} = 0$$

for such geodesic congruences

Second we use the equations of motion to recognize that for null ξ,

$$R_{cd}\xi^c\xi^d = G_{cd}\xi^c\xi^d = 8\pi T_{cd}\xi^c\xi^d \geq 0.$$

This is called the weak energy condition and is very reasonable classically.

Putting this together, we get

$$\frac{d\theta}{d\lambda} \leq -\frac{1}{2}\theta^2$$

so

$$\frac{1}{\theta_f} - \frac{1}{\theta_i} \geq \frac{1}{2}(\lambda_f - \lambda_i).$$

Now notice that this means that θ must always decrease.
But even more dramatically, if $\theta_i = -\theta_0$ at some point, then in ~~at~~ affine parameter $\frac{1}{2\theta_0}$ we have $\theta_f \to -\infty$.
This can happen if geodesics focus and intersect.