QABH, Lecture 7, The Schwarzschild black hole

We now move from a discussion of QFT in curved spacetime to black holes.

The first and most important solution we consider is the Schwarzschild black hole.

This is the metric

$$ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2m}{r}} + r^2 d\Omega^2$$

The first point to recognize is that this metric satisfies the vacuum equations of G.R.

$$R_{\mu\nu} = 0.$$

also $R = 0$.

so except for $r = 0$, there is no matter anywhere!

However, this doesn't mean that the curvature components are 0.

We will see this explicitly later.

To see the physical significance of m, consider the asymptotic form of the geodesic equation,

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 0$$

In the low velocity limit, the only relevant term comes from $\alpha = \beta = t$ and with $\mu = r$, we find

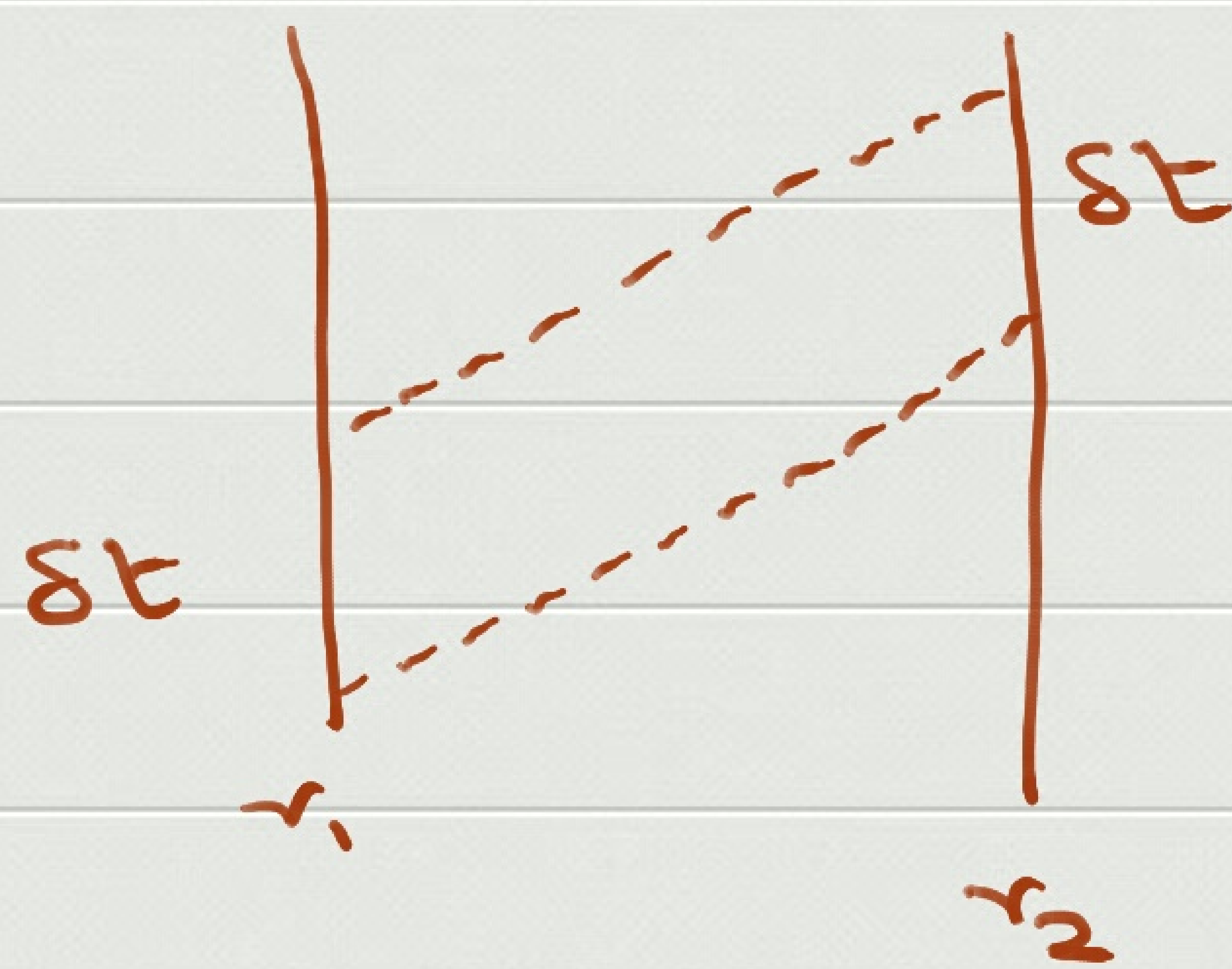
$$\frac{d^2 r}{d\tau^2} + \frac{m}{r^2} = 0$$

$$\Rightarrow \quad \frac{d^2 r}{d\tau^2} = -\frac{m}{r^2} \quad \Rightarrow \quad m = GM.$$

We will consider the geodesic equation again shortly.

Red-shifts

The first significant physical effect we see is that of the red shift. Consider observers at radial positions r_1, r_2.



The person at r_1 is sending light rays to the person at r_2.

The rays travel along

$$dr^2\left(1-\frac{2m}{r}\right)^{-1} - \left(1-\frac{2m}{r}\right)dt^2 = 0$$

but crucially, the coordinate time δt measured by both is equal.

but the proper time is different. For observer 1, the proper time is

$$\delta\tau_1 = \sqrt{\left(1-\frac{2m}{r_1}\right)}\ \delta t$$

and for 2, it is

$$\delta\tau_2 = \sqrt{\left(1-\frac{2m}{r_2}\right)}\ \delta t$$

so

$$\frac{\delta\tau_1}{\delta\tau_2} = \sqrt{\frac{\left(1-\frac{2m}{r_1}\right)}{1-\frac{2m}{r_2}}} \quad \text{or} \quad \frac{\omega_1}{\omega_2} = \sqrt{\frac{\left(1-\frac{2m}{r_2}\right)}{\left(1-\frac{2m}{r_1}\right)}}$$

This means, in particular that if observer 2 is at ∞ we have

$$\omega_2 = \omega_1 \sqrt{1 - \frac{2m}{r_1}}.$$

as $r_1 \to 2m$, we see $\omega_2 \to 0$!

So, it becomes infinitely hard for photons to climb out from $r_1 = 2m$.

To understand what is happening at that spot consider an infalling geodesic.

The action is

$$S = \int \sqrt{\left(1-\frac{2m}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{\left(1-\frac{2m}{r}\right)}}\, d\tau \quad , \quad \dot{t} \equiv \frac{dt}{d\tau} \; ; \; \dot{r} \equiv \frac{dr}{d\tau}$$

So the eoms are

$$\frac{d}{d\tau} \frac{\dot{t}\left(1-\frac{2m}{r}\right)}{L} = 0$$

$$\frac{d}{d\tau}\left(\frac{\dot{r}}{\left(1-\frac{2m}{r}\right)} \Big/ L\right) = \frac{\frac{2m}{r^2}\dot{t}^2 + \frac{\dot{r}^2}{\left(1-\frac{2m}{r}\right)^2}\frac{2m}{r^2}}{L}.$$

The second eq. is harder, but we don't have to solve it separately.

This is because we can fix the reparametrization invariance of the =ns, we set $L = 1$

From the first eq,

$$\dot{t} = \frac{c}{\left(1-\frac{2m}{r}\right)}$$

and $\dfrac{c^2 - \dot{r}^2}{\left(1-\frac{2m}{r}\right)} = 1$, from the constraint

so, $\dot{r}^2 = c^2 - \left(1-\frac{2m}{r}\right).$

If we assume the particle has "0" energy so that very far away as $r \to \infty$, $\dot{t} \to 1$, we set $c=1$.
and

$$\dot{r}^2 = \frac{2m}{r}$$

And we also have

$$\frac{dt}{d\tau} = \frac{1}{1-\frac{2m}{r}}$$

Now the remarkable thing is that if we consider r as a function of τ then the equation is perfectly regular at $r = 2m$.

The solution is just

minus sign because particle is infalling. $\rightarrow \; -\int \sqrt{r}\, dr = \sqrt{2m}\, d\tau$

or $r_0^{3/2} - r^{3/2} = 3\sqrt{\frac{m}{2}}\,(\tau - \tau_0).$

So starting at a finite value of r_0 at $\tau_0 = 0$, the observer reaches $r=0$ in finite time!!

But looking at the t-equation we see something funny happening.

We see that

$$\frac{dt}{d\tau} \to \infty \quad \text{as} \quad r \to 2m.$$

This is indicative of a deep effect

a) The infalling observer passes through the horizon in finite proper time and reaches the singularity in finite proper time

b) For someone watching from infinity, it looks like the observer never crosses the horizon!

[Check (a) =ns of G.R. are satisfied
(b) geodesic equations
(c) simulation.
(d) Riemann tensor.] Mathematica

Even though the region at $r = 2m$ leads to dramatic global effects, locally it is unremarkable.

A measure of the "magnitude" of the curvature is given by $R_{ijkl} R_{\alpha\beta\gamma\delta} g^{\alpha i} g^{\beta j} g^{\gamma k} g^{\delta l} \equiv R^2$

We see that for the Schwarzschild B.H.

$$R^2 = \frac{-24m^2}{r^6}$$

at $r = 2m$, this is $R^2 = -\frac{3}{8m^4}$

This is a dimensionful quantity, but if we have some other length scale l_{ruler}, we can consider

$$l_{ruler}^4 R^2$$

and make it arbitrarily small by taking m large enough.

"LARGE BLACK HOLES HAVE SMOOTH HORIZONS"

This suggests that the bad behaviour of the metric at $r=2m$ is only a coordinate singularity.

Now we explore how this can be removed using an appropriate coordinate redefinition.

The first step is to define the so-called tortoise coordinate so that the metric looks like

$$ds^2 = f(r_*)\left[-dt^2 + dr_*^2\right] + g(r_*)\,d\Omega^2$$

clearly we need,

$$\left(1-\frac{2m}{r}\right) dr_*^2 = \frac{dr^2}{1-\frac{2m}{r}} \quad ; \quad dr_* = \frac{dr}{1-\frac{2m}{r}}$$

which leads to

$$r_* = r + 2m \ln \left| \frac{r-2m}{2m} \right|$$

as

$$r \to 2m, \quad r_* \to -\infty.$$

Next we define

$$U = -e^{\alpha(r_*-t)} \; ; \quad V = e^{\alpha(r_*+t)}$$ will fix α below.

then

$$dU\, dV = -\alpha^2 (dr_* - dt)(dr_* + dt)\, e^{2\alpha r_*}$$

Now near the horizon

$$e^{2\alpha r_*} = e^{2\alpha r_h} \left(\frac{r-2m}{2m} \right)^{4m\alpha}$$

So if we set $4m\alpha = 1$, we see that the near horizon metric becomes proportional to $dU\, dV$.

$$dU\,dV = \left(-1/16M^2\right) r \,(1-2M/r)\cdot\left(\tfrac{1}{2M}\right) e^{+r/2M} \left(dr_*^2 - dt^2\right)$$

More precisely

$$ds^2 = -dU\,dV\left(\frac{32M^3 e^{-r/2M}}{r}\right) + r^2 d\Omega^2$$

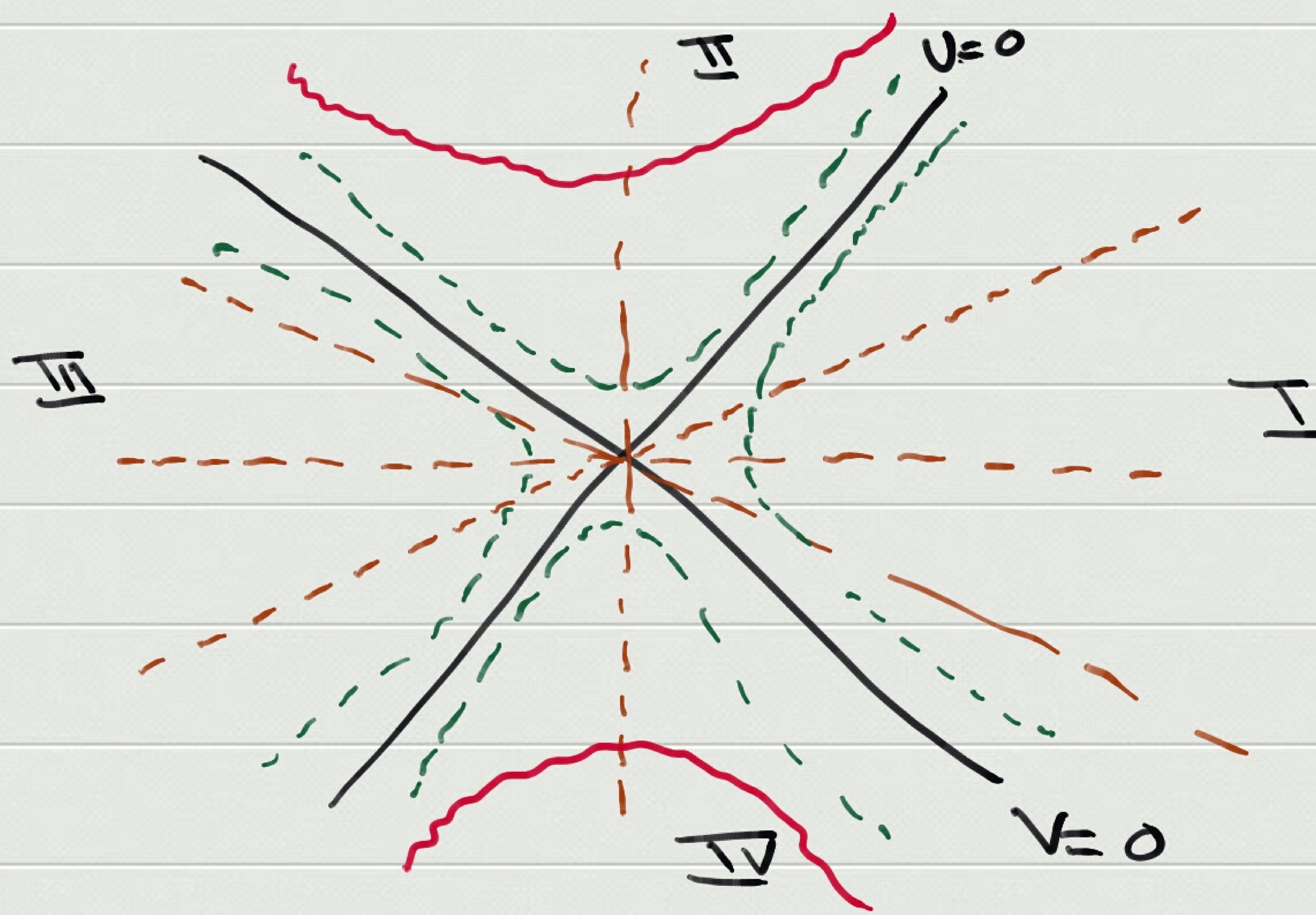
which is manifestly non-singular as $r \to r_h$.

The horizon, in these coordinates is $U = 0$. We can smoothly extend the metric into the region $U > 0$.

There is, however, a true singularity at $r = 0$.

We see this from the fact that curvature invariants blow up. So the geometry does not extend past $r=0$ (or $r_* = 0$).

This new coordinate system gives us a picture of the Kruskal extension of the Schwarzschild black hole.



In regions II $(U>0, V>0)$, we can introduce another patch through $U = e^{\alpha(r_*-t)}$; $V = e^{\alpha(r_*+t)}$; in region III $(U>0, V<0)$, $U = e^{\alpha(r_*-t)}$; $-V = e^{\alpha(r_*+t)}$

In region $\overline{\text{IV}}$, $(U<0,\ V<0)$

$$-U = e^{\alpha(r_* - t)} \ ; \ -V = e^{\alpha(r_* + t)}$$

Note that constant $t \Rightarrow$ constant $\frac{V}{U}$

$r =$ constant $\Rightarrow UV =$ const.

So $r=$const. are hyperboloids in the U, V plane whereas $t =$ const. are straight lines passing through the origin.

The point where $r=0$ is where $r_* = 0$ and $UV = 1$.

This is the singularity.

Away from the singularity, this diagram is reminiscent of Rindler space!