Quantum Aspects of Black Holes, Lec 23, 18 Nov 2016

Today we will try and give a simple and concrete example of black hole complementarity.

[Based on arXiv:1603.02812]

To simplify the situation, we move to anti-de Sitter space. This is the geometry

$$ds^2 = -(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

It satisfies the equations

$$G_{\mu\nu} = 3g_{\mu\nu}$$ i.e. with a negative cosmological constant

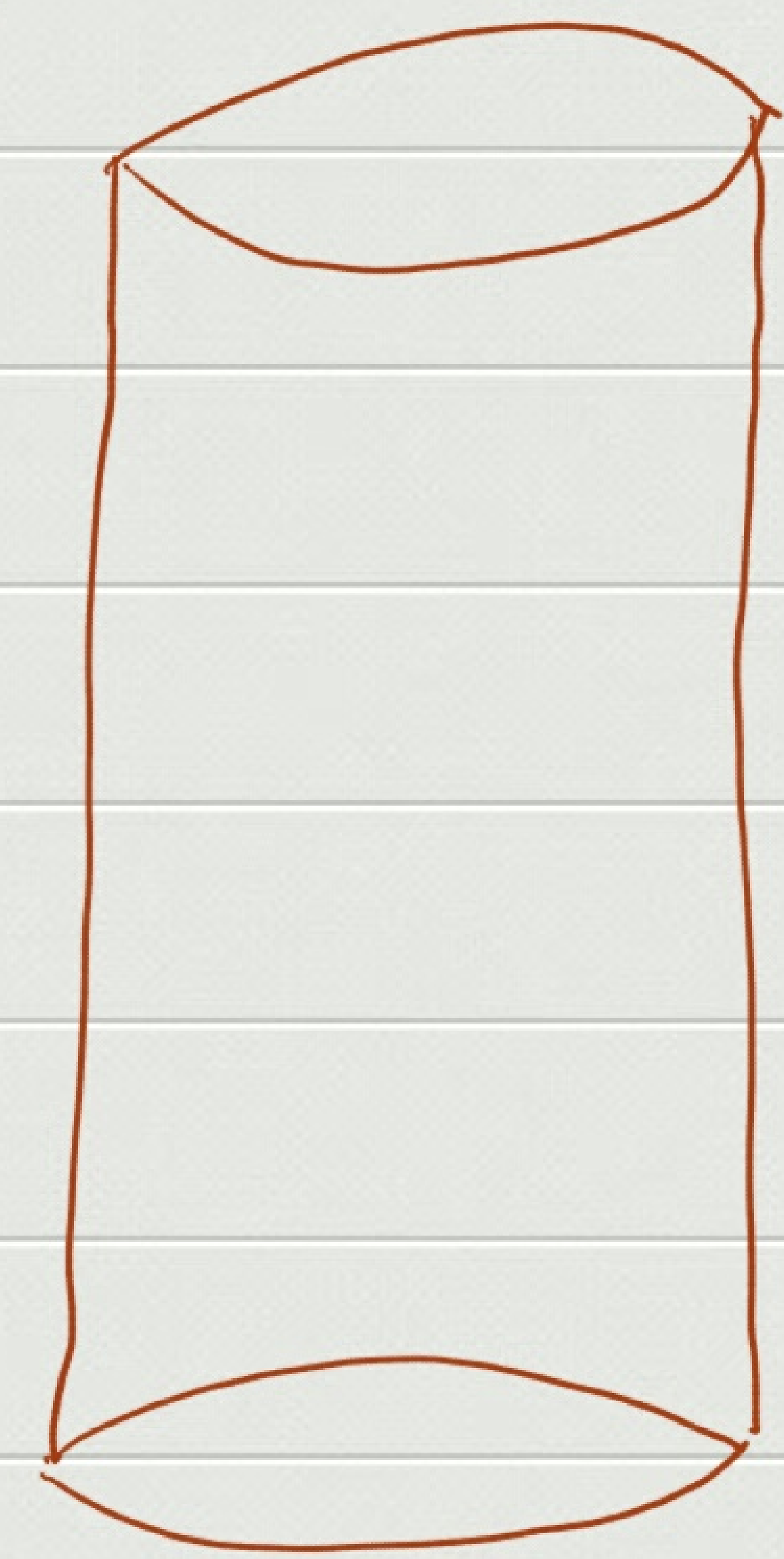
These units correspond to $l_{ads} = 1$.

We also have a dimensionless parameter in the problem

$$\left(\frac{l_{ads}}{l_{Pl}} \right)^4 = N$$

We will be interested in the part of the Hilbert space well below N.

Geometrically, it is convenient to think of AdS as a solid cylinder



Even though the bdry is at "∞", a light ray can go through the bulk to the other side in finite time

The simplest way to see this is to go to "tortoise coordinates"

$$\frac{dr^2}{(1+r^2)} = (1+r^2)\, dr_*^2$$

or

$$r_* = \tan^{-1} r \quad , \quad r_* = \pi/2 \text{ at bdry}$$
$$= 0 \text{ in center}$$

So in a time $\delta t = \pi$, a light ray can cross from bdry to center and to the bdry on the other side.

Let us consider a simple scalar field with mass m that satisfies

$$(\Box + m^2)\Phi = 0$$

Then using our standard techniques of QFT in curved space, we can quantize the field. It turns out that the field can be expanded in modes

$$\varphi_{\ell,n}(r)\, e^{-i\omega_n t}\, Y_{\ell m}(\Omega)$$

where

d for AdS_{d+1}

$$\omega_n = \Delta + \ell + 2n, \qquad m^2 = \Delta(\Delta - 3)$$

The quantization arises because AdS is effectively like a box.

We can the expand a bulk field as

$$\phi(t, r, \Omega) = \sum O_{n,l,m} \, \xi_{n,l}(r) \, e^{-i\omega_n t} \, Y_{l,m}(\Omega) + h.c.$$

The ξ can be written explicitly in terms of hypergeometric functions, but we will not need the explicit expressions.

Now, there is a 1-1 correspondence between these field operators and their boundary values

$$O(t, \Omega) = \lim_{r \to \infty} r^{\Delta} \phi(r, t, \Delta)$$

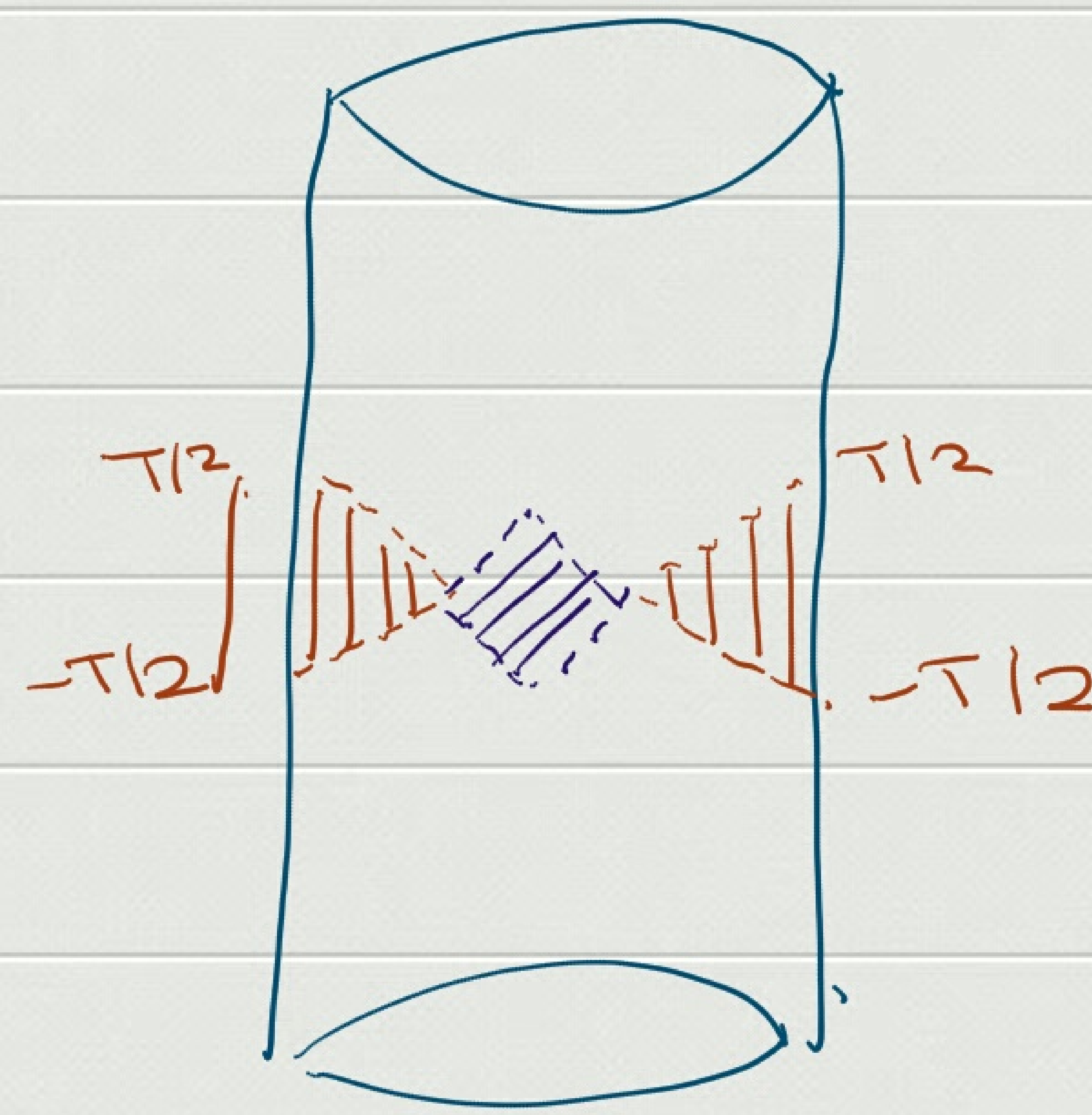
given $O(t, \Omega)$ we recover $O_{n,l,m}$ and therefore ϕ.

We can also think of this in terms of solving a radial equation to obtain ϕ given its boundary value.

Due to the quantization of frequency we need information about $O(t, \Omega)$ on a band of length π.

What if we consider values of O in a range $[-\frac{T}{2}, \frac{T}{2}]$ with $T < \pi$? This corresponds to fields in the range

$$\cot\frac{T}{2} < r < \infty .$$



__1 - 1 correspondence__

$$\left\{ O(t) : \frac{-\pi}{2} < t < \frac{\pi}{2} \right\} \longleftrightarrow \begin{array}{l} \phi(r, \theta, \Omega) : \quad 0 < r < \infty \\ \pi(r, \theta, \Omega) \end{array}$$

$$\left\{ O(t) : -\frac{T}{2} < t < \frac{T}{2} \right\} \longleftrightarrow \begin{array}{l} \phi(r, \theta, \Omega) : \quad \cot \frac{T}{2} < r < \infty \\ \pi(r, \theta, \Omega) \end{array}$$

We will now show that in a theory of gravity $\Phi(r=0, t=0, \Omega)$ can be written as a very complicated polynomial in $O(t, \Omega)$.

This goes beyond the simple statements we made now, which were based on solutions of PDEs.

It is trivially true that

$$\phi(0) = \sum_{ij} c_{ij} \, |\psi_i\rangle\langle\psi_j|$$

center of AdS

where $|\psi_i\rangle$ are Fock space states.
In general single-particle states are created through

$$|n\rangle = O_n^{+} |0\rangle$$

It looks like this requires information about $O(t)$ for a range T.

But we claim that with a suitable choice of $f(t)$ with support only in $[-\frac{T}{2}, \frac{T}{2}]$ we can generate $|n\rangle$!

This is a version of the Reeh-Schlieder theorem.

To do this, we need to write

$$f(t) = e^{-int} + \text{negative frequencies}$$

because we only need to replicate the positive frequencies to get the state right.

Proof: Assume this was not true. Then

$$\exists \, |\psi\rangle, \quad \text{s.t.} \quad \langle 0| \int_{-T/2}^{T/2} O(t)\, f(t)\, |\psi\rangle = 0, \quad \forall \, f(t)$$

$$\Rightarrow \quad \langle 0|\, O(t)\, |\psi\rangle = 0, \quad \forall \; t \in \left[-\frac{T}{2}, \frac{T}{2}\right]$$

but $\quad \sum_E \langle 0 | O(t) | E\rangle \langle E | \psi\rangle.$

$$= \sum_E \left(\langle 0|\, O(0) | E\rangle \langle E | \psi\rangle \right) e^{-iEt}$$

has only positive frequencies. So it can be analytically continued in the lower half t plane.

So $\quad O(t) = 0, \; \forall \; t \in \left[-\frac{T}{2}, \frac{T}{2}\right] \Rightarrow O(t) = 0 \;\; \forall \; t$

$$\Rightarrow |\psi\rangle = 0.$$

Similarly all multi-particle states can also be generated by acting in the region $[-T, T]$.

This means that each $|\psi_i\rangle$ can be written as

$$|\psi_i\rangle = X_{\psi_i} |0\rangle .$$

where X is a simple polynomial in operators living on $[-\frac{T}{2}, \frac{T}{2}]$.

This means that we can write

$$\Phi(0) = \sum_{i,j} C_{ij} \, X_{\psi_i} |0\rangle \langle 0| X^{+}_{\psi_j}$$

so far we have used only QFT.

There is no violation of locality in the formula above.

Now we add some effects that are unique to gravity.

Recall that in gravity, the Hamiltonian is given by a boundary term.

More precisely, we can write

$$H = \int T^{00} \, d\Omega$$

where

$$T^{\mu\nu} = \frac{N^2}{8\pi} \int (\Theta^{\mu\nu} - \Theta \gamma^{\mu\nu}) - \text{subtractions}$$

where

$$\Theta^{\mu\nu} = \nabla^{\mu} n^{\nu}$$ ← outward normal to bdry.

But now we can write

$$|0\rangle\langle 0| = \lim_{\alpha \to \infty} e^{-\alpha H} = P_0$$

In fact we can work with an approximation to

$$P_0 \approx \sum_{p=0}^{p_c} (-1)^p \frac{(\alpha H)^p}{p!}$$

We need to choose α and p_c so that

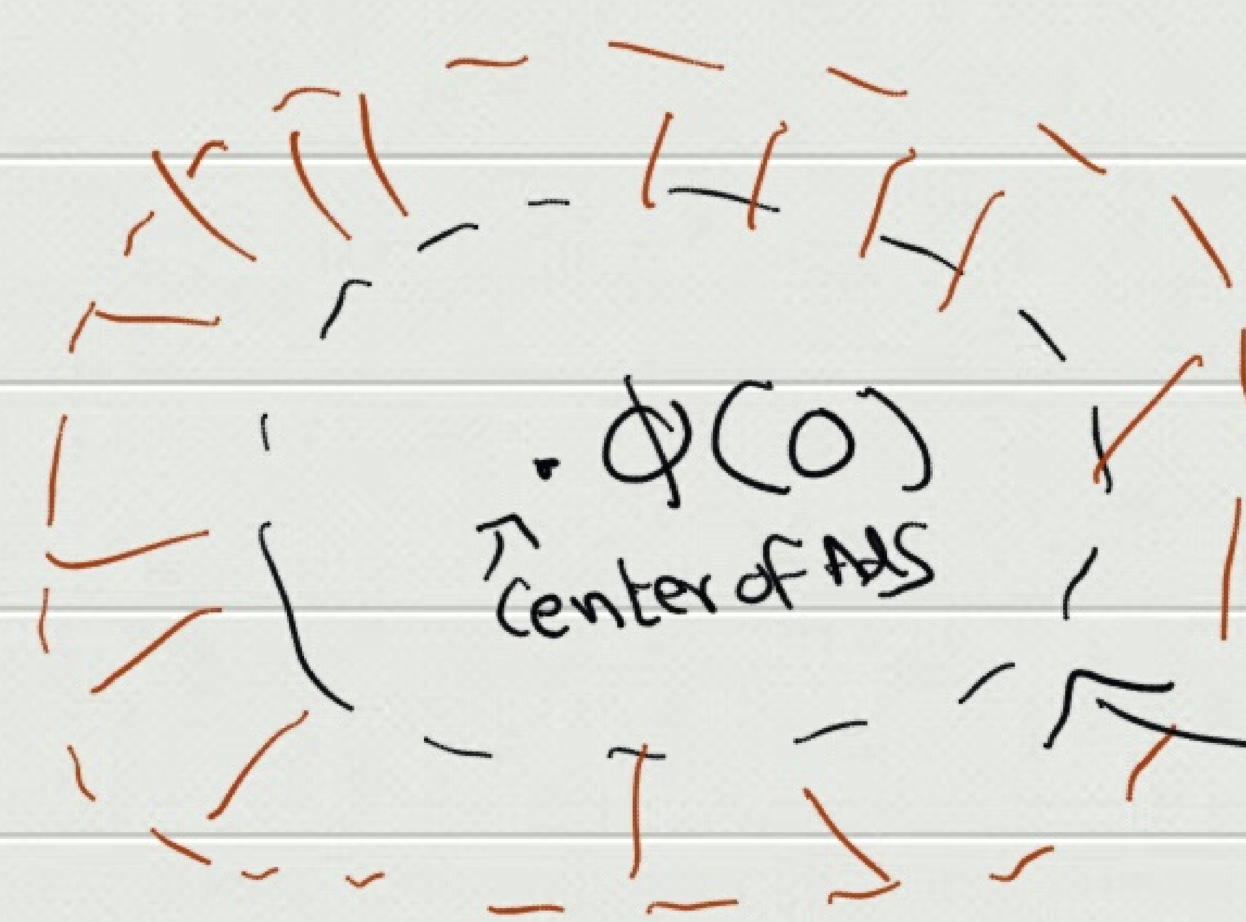
$$e^{-\alpha E_{min}} << 1, \quad p_c >> \alpha E_{max}$$

e.g. $\alpha = \ln(N)$, $p_c = N \ln N$ gives a polynomial Q which well-approximates $|0\rangle\langle 0|$

Then we can finally write

$$\phi(0) = \sum_{i,j} C_{ij} \, X_{\psi_i} \, Q \, . \, X^{\dagger}_{\psi_j}$$

So we have successfully written



Spacelike separated ops in annulus.

$\phi(0)$ in terms of operators uniformly separated spatially from it!