The Black Hole Information Paradox
Assignment 1

Due Saturday, 23 January 2021

1. **Two-point function**
   In the lectures, we argued that in the vicinity of a point where the metric is
$$ds^2 = -d\mathcal{U}d\mathcal{V} + \delta_{\alpha\beta}d\xi^\alpha d\xi^\beta \tag{1}$$
   if one consider two points, x_1, x_2 that are very close to each other then in any smooth state, $|\Psi\rangle$,
$$\langle\Psi|\phi(x_1)\phi(x_2)|\Psi\rangle = \frac{N}{s_{12}^{\frac{d-1}{2}}}, \tag{2}$$
   where s_{12} is the square of the geodesic distance between the points. We expect this to hold provided $\sqrt{s_{12}}$ is smaller than the curvature-scale of the geometry and the length-scale set by the mass of the field but larger than the short-distance cutoff in the theory.

   Consider a *massive free scalar field* in $(d+1)$-dimensional flat space. Verify the result above and find the constant N. Also determine the right $i\epsilon$ prescription in the correlator. Recall that we are considering a Wightman function i.e. we demand that $\phi(x_2)$ is on the right and $\phi(x_1)$ is on the left in the correlator above.

2. **Constants in the delta function**
   In the lecture, we argued that
$$\lim_{\delta V \to 0} \frac{(\delta V)^2}{(-\delta U \delta V + \delta_{\alpha\beta}\delta y^\alpha \delta y^\beta)^{\frac{d+3}{2}}} = K\delta^{d-1}(\delta y^\alpha) \tag{3}$$
   Find the constant K.

3. **Modes across a null surface**
   In the class, we introduced modes $\mathfrak{a}$ and $\widetilde{\mathfrak{a}}$ and argued that, in any smooth state $|\Psi\rangle$ we have
$$\langle\Psi|\mathfrak{a}\widetilde{\mathfrak{a}}|\Psi\rangle = \frac{e^{-\pi\omega_0}}{1-e^{-2\pi\omega_0}} \tag{4}$$
   Show that we must also have
$$\langle\Psi|\mathfrak{a}\mathfrak{a}^\dagger|\Psi\rangle = \langle\Psi|\widetilde{\mathfrak{a}}\widetilde{\mathfrak{a}}^\dagger|\Psi\rangle = \frac{1}{1-e^{-2\pi\omega_0}} \tag{5}$$
   In addition show that
$$\begin{aligned}&[\mathfrak{a}, \widetilde{\mathfrak{a}}] = 0;\\ &[\mathfrak{a}, \mathfrak{a}^\dagger] = [\widetilde{\mathfrak{a}}, \widetilde{\mathfrak{a}}^\dagger] = 1.\end{aligned} \tag{6}$$
   Notice the equation (6) provides an operator relation, whereas equation (5) tells us only about a two-point correlator.

4. **Numbers in Quantum Gravity**

(a) In the lectures next week, we will derive a formula for the temperature of a Schwarzschild black hole with mass M and find that it is $T = \frac{1}{8\pi M}$. Restore factors of $\hbar, c, G$ and work out the temperature of black holes (in degrees Kelvin) with (a) the mass of the sun (b) the mass of the Earth and (c) the mass of a large ship, 100,000 tons.

(b) Find the Unruh temperature for an observer who accelerates at the acceleration due to gravity, $g = 9.81 m/s^2$. What is the acceleration required to observe an Unruh temperature of $1K$.