The Black Hole Information Paradox
Assignment 2

Due Tuesday, 2 February 2021

1. **Wave equation about black holes**
   Consider a minimally coupled scalar field that satisfies the equation

$$(\Box - m^2)\phi = 0 \tag{1}$$

   a) In the geometry of the Schwarzschild black hole, expand this equation in the r_*, t, Ω coordinates, and show that it reduces to

$$\frac{1}{f(r)r^{d-1}}\partial_{r_*} r^{d-1}\partial_{r_*}\phi - \frac{1}{f(r)}\partial_t^2\phi + \frac{1}{r^2}\Box_\Omega\phi - m^2\phi = 0, \tag{2}$$

   where $\Box_\Omega$ is the Laplacian on the sphere.

   Consider an ansatz for the solution of the form

$$\phi_{\omega,\ell}(t, r_*, \Omega) = c(r)\chi_{\omega,\ell}(r_*)e^{-i\omega t}Y_\ell(\Omega). \tag{3}$$

   b) Find a choice of $c(r)$ so the the differential equation for $\chi_{\omega,\ell}$ becomes

$$\frac{\partial^2\chi_{\omega,\ell}(r_*)}{\partial {r_*}^2} + \omega^2\chi_{\omega,\ell}(r_*) = V(r_*)\chi_{\omega,\ell}(r_*), \tag{4}$$

   Find the form of $V(r_*)$.

   This analysis is valid for black holes in AdS and in flat space and also for higher-dimensional black holes. In the manipulations above, do not hesitate to move between r, understood as a function of r_*, and r_* itself. You will not need any details of this relation except for the fact that $\frac{dr}{dr_*} = f(r)$.

2. **Greybody factors in flat space**
   In this problem we specialize to a *massless* field for simplicity i.e. we put $m^2 = 0$ in the analysis of Problem 1.

   In flat space, we considered two kinds of solutions of the wave equation. Near the horizon, these had the form

$$\begin{aligned} f^{\text{in}}(\omega, \ell, r_*) &\underset{r\to r_h^+}{\longrightarrow} \mathfrak{h}_{\omega,\ell}e^{-i\omega r_*}; \\ f^{\text{out}}(\omega, \ell, r_*) &\underset{r\to r_h^+}{\longrightarrow} e^{i\omega r_*} + \mathfrak{g}_{\omega,\ell}e^{-i\omega r_*}. \end{aligned} \tag{5}$$

   a) Using the canonical commutation relations near the horizon, show that $|\mathfrak{h}_{\omega,\ell}|^2$ and $|\mathfrak{g}_{\omega,\ell}|^2$ satisfy a linear constraint. Find this constraint.

These solutions are chosen so that near $r \to \infty$ they have the form

$$\begin{aligned} f^{\text{in}}(\omega, \ell, r_*) &\underset{r\to\infty}{\longrightarrow} r^{\frac{1-d}{2}} \left(e^{-i\omega r_*} + \widetilde{\mathfrak{h}}_{\omega,\ell} e^{i\omega r_*} \right); \\ f^{\text{out}}(\omega, \ell, r_*) &\underset{r\to\infty}{\longrightarrow} r^{\frac{1-d}{2}} \widetilde{\mathfrak{g}}_{\omega,\ell} e^{i\omega r_*} \end{aligned} \tag{6}$$

b) Show that $\widetilde{\mathfrak{g}}_{\omega,\ell}$ is proportional to $\mathfrak{h}_{\omega,\ell}$ and find the constant of proportionality.

[*Hint:* You should not need to solve the wave equation for this. Simply note that when the wave equation is written in the form (4), the Wronskian between two independent solutions is conserved.]

The quantity $\widetilde{\mathfrak{g}}_{\omega,\ell}$ is of physical interest because it tells us how much of the "outgoing" mode escapes to infinity.

3. **Solving the wave equation**
In the lectures, we repeatedly alluded to the fact that one could specify the behaviour of a solution near the horizon, and then solve away from the horizon systematically. In particular, this is how we specified the modes behind the horizon, which we denoted by $\widetilde{f^{\text{out}}}(\omega, \ell, r_*)$. In this problem, we will explore how this is done.

   a) First, let us consider the "potential" in (4). Show that we can expand this potential in the form
$$V_{\text{hor}}(r_*) = \sum_m V_m e^{\frac{4\pi m r_*}{\beta}}, \tag{7}$$
   Show that such an expansion is valid close enough to the horizon ($r_* \to -\infty$) and only runs over $m > 0$. You do not need to find the coefficients V_m explicitly but only argue that the series exists.

Now, the idea is to write a solution to the equation (4) in the form

$$\chi_{\omega,\ell}(r_*) = \sum_{n=0}^{\infty} \chi^{(n)}_{\omega,\ell}(r_*). \tag{8}$$

with

$$\chi^{(n)}_{\omega,\ell}(r_*) = \int_{-\infty}^{r_*} \chi^{(n-1)}_{\omega,\ell}(r'_*) V_{\text{hor}}(r'_*) G_{\text{hor}}(r'_*, r_*) dr'_*, \tag{9}$$

and

$$G_{\text{hor}}(r'_*, r_*) = \frac{1}{\omega} \sin(\omega(r_* - r'_*))\theta(r_* - r'_*). \tag{10}$$

This series can be seeded with

$$\chi^{(0)}_{\omega,\ell}(r_*) = e^{-i\omega r_*} \tag{11}$$

   b) Show that the series (8) leads to a solution of the wave equation in the regime where (7) holds. Note that at each step we only use the solution from the previous step. And at each step, to find the solution at a value of r_*, we only need the previous solution from the horizon up to r_*, and not beyond it.

   c) Using the expansion above, find an expression for $\chi^{(1)}_{\omega,\ell}$. Again, the expression can involve a sum over V_m.

4. **Evaporation Times**

The precise rate of decay of a black hole depends on the greybody factors. In this problem, these factors will not be important. Let us assume, for simplicity, that a four dimensional black hole loses mass at the rate

$$\frac{dM}{dt} = -\sigma T^4 A \tag{12}$$

where A is its area, T its temperature and σ is the Stefan-Boltzmann constant.

Find the evaporation time for a black hole with (a) the mass of the sun (b) the mass of the Earth and (c) the mass of a large ship, 100,000 tons.