The Black Hole Information Paradox
Assignment 4

Due Saturday, 20 March 2021

1. **Fourier transforms and the Reeh Schlieder theorem**
   Let $f(t)$ be a periodic function on $[-\pi, \pi]$ with the Fourier expansion.

$$f(t) = \sum_{n=-\infty}^{\infty} f_n e^{-int}.$$

   (a) Show that we can find $g(t)$ with support only in $t \in [-\epsilon, \epsilon]$ with the property that the *positive* Fourier coefficients of $g(t)$ approximate those of $f(t)$ arbitrarily well. More precisely, $\forall \delta > 0$ show that we can ensure

$$r = \sum_{n=1}^{\infty} \left| \int_{-\epsilon}^{\epsilon} g(t) e^{int} \frac{dt}{2\pi} - f_n \right|^2 < \delta. \tag{1}$$

   (b) Take $\epsilon = 1$ and $f(t) = e^{-it}$. (You can choose a different function and value of ϵ if you prefer.) Choose a complete basis of functions $b_q(t)$ for the range $[-\epsilon, \epsilon]$ labeled by $q = 1, \ldots \infty$. Write a numerical script that finds a sequence of approximations

$$g_{q_0}(t) = \sum_{q=1}^{q_0} a_q b_q(t), \tag{2}$$

   by minimizing the residual

$$r_{q_0} = \sum_{n=1}^{\infty} \left| \int_{-\epsilon}^{\epsilon} g_{q_0}(t) e^{int} \frac{dt}{2\pi} - f_n \right|^2, \tag{3}$$

   with respect to the coefficients a_q. Plot the minimum value as a function of q_0.

Note:

(a) The relevance of this problem to the Reeh-Schlieder theorem is as follows. Consider the state $\int O(t) f(t) |0\rangle$. Only the positive Fourier components of f are relevant since the energy of the vacuum cannot be lowered. So to approximate this state by another state $\int O(t) g(t) |0\rangle$ where $g(t)$ has support only on a small interval, we only need to ensure that the positive Fourier components of $f(t)$ and $g(t)$ match. The norm on states is more complicated than the residual that appears in Equation (1) since it involves the two-point function of $O(t)$. This creates some additional subtleties, but the idea is the same.

(b) Note that the residual in Equation (3) involves, in principle, a sum over high Fourier modes of $g(t)$. To ensure that these modes do not contribute, you should choose a basis of functions whose high Fourier modes die off very rapidly. The numerical script distributed with the notes from Lecture 12 may be useful.

2. **Locality and the Reeh-Schlieder theorem**
Let $\phi(t, r, \Omega)$ be a minimally coupled scalar field dual to an operator of dimension Δ in global AdS_{d+1}. (See Appendix B of the review.) Assume that gravity has been *switched off.* Consider the operator

$$X = \int d^{d-1}\Omega \int_0^{r_0} dr \phi(t=0, r, \Omega) f(r, \Omega), \tag{4}$$

which is a local operator obtained by smearing the scalar field over the radial range $[0, r_0]$ and over the entire S^{d-1} with the weight function $f(r, \Omega)$. The Reeh-Schlieder theorem tells us that we can find a function $g(t, \Omega)$ with support in $[0, \epsilon]$ so that the operator Y defined as

$$Y = \lim_{r \to \infty} r^{\Delta} \int d^{d-1}\Omega \int_{-\epsilon}^{\epsilon} dt \phi(t, r, \Omega) g(t, \Omega) \tag{5}$$

has the property that

$$\| X|0\rangle - Y|0\rangle \| \tag{6}$$

can be made as small as desired.

(a) Write down a mode expansion for the commutator

$$[X, Y],$$

in terms of the functions f and g. You can use either the mode expansion of Appendix B or use the explicit position-space two-point function in AdS to obtain this expression.

(b) Prove that $[X, Y] = 0$ provided that $r_0 < \cot \epsilon$.

Note: We emphasized in the lectures that the Reeh-Schlieder theorem holds even without gravity. Your proof above is a check that the Reeh-Schlieder theorem is perfectly consistent with microcausality.

The results above tell us that $\langle 0|e^{-iX} Y e^{iX}|0\rangle = \langle 0|Y|0\rangle$. So even though the action of Y on the vacuum is the same as the action of X, the action of the unitary e^{iX} is completely invisible to Y.

3. **Using non-vacuum base states.**
Let $|\Psi\rangle$ be a *finite* superposition of arbitrary energy eigenstates of a quantum field theory in AdS. Show that the action of the algebra of operators from the time band $[0, \epsilon]$ on $|\Psi\rangle$ is also dense in the Hilbert space.
Note: We have used the vacuum as a base state in our discussion. But the result above tells us that we can replace the vacuum with any finite-energy state.