# The Black Hole Information Paradox
# Assignment 5

*Due Saturday, 20 March 2021*

1. **Detecting excitations in AdS**
   Let $\phi$ be a minimally coupled scalar field in global $\mathrm{AdS}_{d+1}$ dual to an operator of dimension $\Delta$. (Here we are using the notational conventions of Appendix B of the review.) Consider the state

$$|f\rangle = e^{i\lambda \int dr d^{d-1}\Omega f(r,\Omega)\phi(r,t=0,\Omega) d^{d-1}\Omega}|0\rangle. \tag{1}$$

   In the exponent of the unitary operator above, we have smeared the field on the $t = 0$ Cauchy slice with a function $f(r, \Omega)$, which we take to have compact support in $r$.

   (a) If $H$ is the Hamiltonian of the field, work out the two point function

$$\lim_{r\to\infty} r^{\Delta}\langle f|H\phi(r,\tau,\Omega')|f\rangle \tag{2}$$

   to *first order* in $\lambda$. Note that, in the correlator, the field insertion is made near the boundary and so the correlator has been scaled by an appropriate power of $r$ to make it finite.

   (b) Show that knowledge of this two-point function for the range $\tau \in [0, \epsilon]$ and for all $\Omega'$ is enough to uniquely fix $f(r, \Omega)$.

   **Note:** In a theory of gravity, the Hamiltonian is itself accessible near the boundary and so the correlator that appears in (2) is accessible near the boundary. The result you prove above shows that for simple states of the form (1), a two-point correlator is sufficient to reconstruct the state from observations in a small time-band near the boundary.

2. **Two point correlator at $\mathcal{I}^+$**
   Consider a free scalar field written in retarded Bondi coordinates, $\phi(u, r, \Omega)$ in Minkowski space. Show that the vacuum expectation value of the two-point function has the following limit at $\mathcal{I}^+$.

$$\lim_{r\to\infty} r^2\langle\partial_u\phi(u,r,\Omega)\phi(u',r,\Omega')\rangle = K\frac{1}{u-u'-i\epsilon}\delta^2(\Omega,\Omega'), \tag{3}$$

   and find the constant $K$.
   **Note:** This two-point function was used in the lectures when we discussed low-energy tests of the principle of holography of information.

3. **Operators in the toy monogamy paradox**
   Let $|0\rangle$ be the global AdS vacuum. Let $B$ be a Hermitian operator with bounded

norm and define $|B\rangle = B|0\rangle$. Let $Q$ be some other operator with the property that $Q|0\rangle = |B\rangle$ and let $P_0 = |0\rangle\langle 0|$. Show that the operator

$$C = \frac{\langle 0|Q^\dagger Q|0\rangle \left(QP_0 + P_0 Q^\dagger - \langle 0|Q|0\rangle P_0\right) - \langle 0|Q|0\rangle Q P_0 Q^\dagger}{\langle 0|Q^\dagger Q|0\rangle - \langle 0|Q|0\rangle^2}, \tag{4}$$

satisfies

$$C|\Psi\rangle = |B\rangle, \tag{5}$$

and find its operator norm, $\|C\|$.

**Note:** These are the operators used to set up a toy version of the monogamy paradox. Even in a nongravitational theory, we can find an operator $Q$ near the boundary so that $Q|0\rangle = |B\rangle$. But such an operator cannot be used as part of a CHSH operator because its operator norm is, in general, not smaller than 1. The use of the vacuum projector allows us to construct another operator $C$, which still satisfies (5) but also has small operator norm.