

18 Feb 2021

Lecture 12: Holography of Info in AdS

The result we proved yesterday can be stated as follows. Let

A_Σ be the algebra of asymptotic operators in the time-band Σ_0, Σ_1 .
(i.e. we smear operators, and take all possible products and linear combinations.)

Then

$$H = A_\Sigma |0\rangle$$

generates the entire Hilbert space (or at least a dynamically closed superselection sector.)

Lets consider this result in two limits

First consider Free-field theory in the bulk.
Then we can check this numerically.

Consider some state, say

$$|n, \lambda\rangle = a_{n, \lambda}^{\dagger} |0\rangle$$

Take a basis of Functions with compact support in $[0, \varepsilon]$: call elements of this basis $\tilde{f}_q$

Then consider the residual

$$|n, \lambda\rangle - \sum_{q=1}^{q_{\max}} c_q \phi(\tilde{f}_q) |0\rangle \|^2$$

and minimize with respect to c_q .

This must be a monotonically decreasing
fn of $q_{\max}$, and our proof shows
this goes to 0 as $q_{\max} \rightarrow \infty$

In the numerical code (shown), we consider
the window fn

$$w(t) = e^{\frac{1}{t(t-1)}} , \quad 0 < t < 1$$

the basis fns are 0 , otherwise

$$\tilde{f}_{q,\alpha}(t, \alpha) = e^{iq_t} \gamma_\alpha(\alpha) w(t)$$

We explicitly see the residual dropping to
0.

See attached code.

Another angle to think about this result is from an AdS/CFT perspective

From that perspective, the result is almost trivial. If we take a light CFT operator and consider

$$O(\tau, \vec{x}) \dots O(\tau_n, \vec{x}) |0\rangle, \quad \tau_i \in [0, \varepsilon]$$

we expect to generate the full H-space (or a dynamically closed superselection sector) because the OPEs of O generate heavier and heavier primaries whose action on the vacuum generates the H-space.

This result can be thought of as a manifestation of entanglement between the near-boundary and bulk region.

Consider a simple qubit analogue of this result. Take

$$|EPR\rangle = a|00\rangle + b|11\rangle.$$

Clearly the Hilbert space is h -dim.

But acting on only the first qubit we can generate all four states.

$$(1 + \sigma^3) |EPR\rangle = 2a |00\rangle$$

$$(\sigma^x + i\sigma^y) |EPR\rangle = 2a |10\rangle$$

$$(1 - \sigma^3) |EPR\rangle = 2b |11\rangle$$

$$(\sigma^x - i\sigma^y) |EPR\rangle = 2b |01\rangle$$

An important point

THIS INVOLVES NO VIOLATION OF LOCALITY!

The reason is that an observer with access to the first qubit can only act with unitary operators!

We use "act" to denote the mathematical

application of a Hermitian operator the state but in general

$$|\psi\rangle \rightarrow X_1 |\psi\rangle$$

is not an allowed transformation of state for someone with access to the first qubit

So someone with access to only the first qubit cannot "create" an excitation for an observer of the second qubit.

2) Likewise an observer on the first qubit cannot obtain any "information" about the second qubit.

Say we transform the state

$$|\psi\rangle \rightarrow U_2 |\psi\rangle$$

↑
unitary on
2nd qubit.

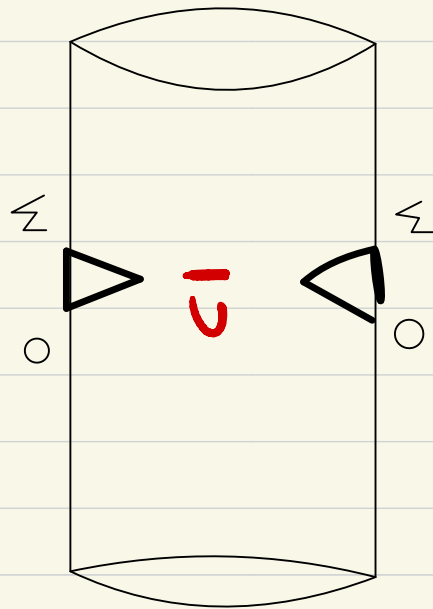
then

$$\langle \psi | U_2^\dagger X_1 U_2 | \psi \rangle = \langle \psi | X_1 | \psi \rangle$$

↑
any observable
in first qubit

Likewise, our theorem in AdS holds also in LQFTs and implies no violation of locality.

In a LQFT (not in gravity as we will see!) there are bulk unitaries that commute with all operators in the time band



A unitary in the red region commutes with all

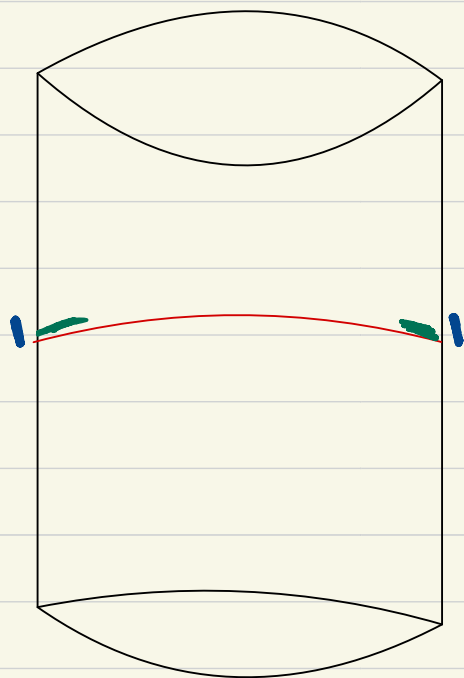
operators from the time band by
microcausality.

Holography of Information

We now show the remarkable result that

"In a theory of Q.G. a copy of all information available on a Cauchy slice is also available near the boundary of the slice. This redundancy is visible in the low energy theory."

First let us make "near the boundary" precise for AdS.



We want to claim all info is on the green parts of the slice. In gravity this is hard to make precise since the bulk fluctuates

So we will instead prove that all info is available on a band $[0, \epsilon]$ at $r \rightarrow \infty$.

These are "morally" the same but the precise result involves the time band!

What this means is:

a) If two states are distinct:

$$|\psi_1\rangle \neq |\psi_2\rangle$$

then $\exists A \in \text{algebra of } [0, \varepsilon]$ s.t.

$$\langle \psi_1 | A | \psi_1 \rangle \neq \langle \psi_2 | A | \psi_2 \rangle.$$

or equivalently

b) All operators in the theory can be approximated arbitrarily well by operators from the algebra $[0, \varepsilon]$

Some comments

- a) This is **much stronger** than the R.S. result we proved earlier.

holography of info $\gg$ Reeh-Schlieder

Earlier we only stated all states could be obtained as $X_{[0,\varepsilon]}|0\rangle$.

Now we are saying all operators have a representation as $X_{[0,\varepsilon]}$

- b) From the perspective of AdS/CFT, this result is trivial. But we are not invoking AdS/CFT here.
This can also be thought of as an explanation of one important aspect of AdS/CFT.

c) This result is eventually a warmup for flat space.

Hamiltonian as a boundary term.

The main physical result we will use that is unique to gravity is that the **energy can be measured** from infinity!

We will consider 3 perspectives on this.

Perspective 1: Gauss Law

A basic feature of gravity is that the total energy can be measured from infinity.

In Ads, we allow the metric to fluctuate in subleading components

$$g_{\mu\nu} = g_{\mu\nu}^{\text{Ads}} + h_{\mu\nu}$$

It is simplest to choose the gauge $h_{r\mu} = 0$ and then boundary conditions on h are

$$h_{ij} \xrightarrow{r \rightarrow \infty} \frac{1}{r^{d-2}}$$

The energy is then

$$H = \frac{d}{16\pi G} \lim_{r \rightarrow \infty} \int d^{d-1} h_{tt} r^{d-2}$$

See hep-th/9902121
- and
gr-qc/9501014
for equivalent
expressions]

Perspective 2 : invariance under small diffs

Say we specify the state of the system on a given time slice using a wavefunctional of the fields [black slice]



We can consider a small diffeomorphism that pushes the slice forward in time in the bulk while not affecting the asymptotic boundary.

The invariance of the wavefunction under such transformations leads to the Wheeler de Witt equation

$$H(x)|\Psi\rangle = 0$$

where $\mathcal{H}(x)$ is the local Hamiltonian density!

So in the canonical approach to gravity, the Hamiltonian, when evaluated, on solutions of the wdw equation is a boundary term.

$$\begin{array}{ccc} \mathcal{H} & |\psi\rangle & = & \mathcal{H}_{\text{bdry}} |\psi\rangle \\ \uparrow & & & \uparrow \\ \text{Hamiltonian} & & & \text{boundary term} \end{array}$$

because the bulk contribution vanishes.

Perspective 3: AdS/CFT

In AdS/CFT the "extrapolate dictionary" tells us that limits of bulk operators are boundary operators.

The metric is dual to the stress tensor.
and

$$\lim_{r \rightarrow \infty} r^{d-2} h_{00} \propto T_{00}$$

clearly therefore the Hamiltonian

$$H = \int T_{00}$$

is the integrated bulk metric near the boundary.

Projector on the vacuum

Since H is an asymptotic operator we can now ask

"what if we measure the energy"

By repeated measurements of the energy we can determine the probability that we get 0.

By the Born rule this is just

$$\langle \psi | P_0 | \psi \rangle$$

in the state $|\psi\rangle$ where P_0 is the projector on the vacuum

$$P_0 = |0\rangle\langle 0|$$

So

P_0 is an asymptotic observable

This is a remarkable property of gravity. In no other theory is a projector on a single state part of the asymptotic algebra.

For instance, in gauge theories we can project onto states of 0 charge. But this is an infinite dimensional space.

Putting everything together

Say we have two distinct states $|\psi_1\rangle$ and $|\psi_2\rangle$.

There must be some operator O that distinguishes between them.

We will show that O can be represented in the asymptotic algebra, A_∞

First, we can always write

$$O = \sum_{n,m} c_{nm} |n\rangle \langle m|$$

where n, m run over some basis.

But recall

$$|n\rangle = X_n |0\rangle$$

$$\langle m| = \langle 0| X_m^\dagger$$

for some $X_n, X_m \in A_\Sigma$

so

$$0 = \sum c_{nm} X_n |0\rangle \langle 0| X_m^\dagger$$

← so far, this formula holds even in a QFT without gravity

But now note

$$|0\rangle \langle 0| = P_0$$

so

$$0 = \sum c_{nm} X_n P_0 X_m^\dagger$$

But

$$x_n \in A_\Sigma$$

$$x_m \in A_\Sigma$$

$$p_0 \in A_\Sigma$$

so

$$0 \in A_\Sigma$$

[algebra is closed under products and
linear combinations.]

Comments

- 1) From the pt. of view of AdS/CFT, this is trivial.

In AdS/CFT, we do not even need a time band of size $[0, \epsilon]$ but a single slice suffices.

This can be thought of as an explanation for **why gravitational theories are holographic**

- 2) Note these results do not hold in non-gravitational theories or even in classical gravity [Explain more in lecture.]