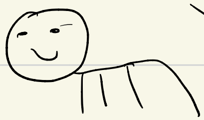
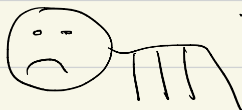
28 Jan 2021

Lecture 6

How close are pure states to mixed states

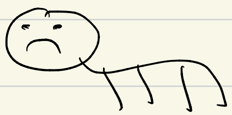
There are two ways we can have uncertainty in quantum mechanics

One is by taking a superposition of two states

$a_1 \, |$  $\rangle + a_2 \, |$  $\rangle$

Other is to have a classical mixture

prob $|a_1|^2$ 

prob $|a_2|^2$ 

In general these states are distinguishable.

If two states are 1 and 2, the expectation value of "A" in

$$|\psi\rangle = a_1|1\rangle + a_2|2\rangle$$

is

$$|a_1|^2 \langle 1|A|1\rangle + |a_2|^2 \langle 2|A|2\rangle$$

$$+ \underbrace{a_2^* a_1 \langle 2|A|1\rangle + a_2 a_1^* \langle 1|A|2\rangle}_{\text{cross-terms}}$$

In the other case, where we have a classical mixture, we have

$$|a_1|^2 \langle 1|A|1\rangle + |a_2|^2 \langle 2|A|2\rangle.$$

The classical mixture is represented by

a density matrix

$$\rho = |a_1|^2 |1\rangle\langle 1| + |a_2|^2 |2\rangle\langle 2|$$

Expectation values are computed by

$$\langle A\rangle_\rho = tr(\rho A)$$

A pure state can also be represented as a special kind of density matrix with

$$\rho^2 = \rho .$$

Say we have a system, with some number of energy levels

$$E_0 - \Delta \leq E_i \leq E_0 + \Delta$$

Then a general superposition of states is

$$\sum a_i |E_i\rangle$$

with some coefficients a_i

On the other hand, one can also consider the microcanonical ensemble.

Here we imagine that the system has equal probability of being in any given energy eigenstate and average over possible outcomes.

This is represented by a microcanonical density matrix:

$$\underbrace{\frac{1}{W}}_{\text{no. of states between } E_0-\Delta, \text{ to } E_0+\Delta.} \sum_i |E_i\rangle\langle E_i|$$

$W = e^S$

We want to ask how "close" "typical states" are to this mixed state.

We need to understand the meaning of

1) "close"

2) "typical states"

Meaning of "close"

The pure state $\sum a_i |E_i\rangle$ and the mixed state $\frac{1}{W}\sum |E_i\rangle\langle E_i|$ are obviously orthogonal to one another.

We want a physical notion of closeness.

$\boxed{\text{"From the pov of physical observations, how easy/difficult is it to distinguish the two cases"}}$

In q.m., observations have to do with the probabilities of various possible outcomes.

If P is a projector on the Hilbert space, i.e. $P^2 = P$ then the main observations we are interested in are

$$\langle \psi | P | \psi \rangle.$$

For instance if P appears in the spectral decomposition of another observable

$$O = \lambda P + \sum \lambda_i P_i$$

then $\langle\psi|P|\psi\rangle$ tells us the probability of obtaining λ upon measuring O in $|\psi\rangle$.

Note the quantum information actually lies in

$$\langle\psi|P|\psi\rangle$$

and not in λ.

We could always choose to measure

$$O \longrightarrow 1000\,O$$

which would take

$$\lambda \rightarrow 1000\lambda$$

but probabilities are unaffected by such changes in convention.

So given some projector, we want to see how

$$\langle \psi | P | \psi \rangle \quad , \quad |\psi\rangle \equiv \text{typical pure state}$$

compares with

$$\mathrm{tr}(\rho P) \quad , \quad \rho \equiv \text{micro density matrix}$$

Next, we need to address the meaning of "typical"

Consider the structure of the Hilbert space.

We can pick any state

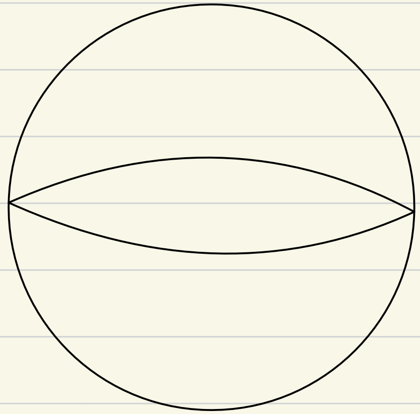
$$\sum a_i |E_i\rangle$$

subject to

$$\sum |a_i|^2 = 1.$$

The Hilbert space is a copy of $\mathbb{C}P^{w-1}$

but for simplicity (and at the cost of some precision) one can visualize it as a sphere.



This is not completely precise since we must also identify

$$|\psi\rangle \to e^{i\theta}|\psi\rangle$$

$$a_i \to e^{i\theta} a_i$$

but this issue will not be crucial below.

The question is: if one picks a point at "random" on the sphere, this gives a state $|\psi\rangle$. What do we expect for $\langle\psi|P|\psi\rangle$ for such a state.

We introduce a probability distribution on the Hilbert space

a_i are complex numbers

$$d\mu_\psi = \frac{1}{V} \delta\left(\sum |a_i|^2 - 1\right) \prod_i d^2 a_i$$

We then ask about the expectation value and higher moments of

$$\delta = \langle\psi| P |\psi\rangle - \mathrm{tr}(\rho \cdot P)$$

w.r.t. to this distribution

Here, V is chosen so that

$$\int d\mu_\psi = 1$$

$$V = \frac{2\pi^w}{\Gamma(w)}$$

So the first quantity to compute is

$\langle S \rangle$

Note

$$\langle \psi | P | \psi \rangle$$

$$= \sum_i |a_i|^2 \langle E_i | P | E_i \rangle$$

$$+ \sum_{i \neq j} a_i a_j^* \langle E_j | P | E_i \rangle.$$

So to compute

$$\int \langle\psi|A|\psi\rangle \, d\mu_\psi$$

We need

$$\int |a_i|^2 \, d\mu_\psi$$

and

$$\int a_i \, a_j^* \, d\mu_\psi$$

Note

$$\sum |a_i|^2 = 1$$ and $\langle |a_i|^2 \rangle$ cannot depend on i. So it must be that

$$\int |a_i|^2 \, d\mu_\psi = \frac{1}{w}$$

Clearly also

$$\int a_i a_j^* \, d\mu_\psi = 0 \quad \text{if } i \neq j$$

so

$$\int \langle \psi | P | \psi \rangle \, d\mu_\psi = \frac{1}{W} \sum \langle E_i | P | E_i \rangle$$

$$= \operatorname{tr}(\rho P)$$

so $\langle \delta \rangle = 0$!

We can also compute $\langle \delta^2 \rangle$

$$\int \left(\langle \psi | P | \psi \rangle - \mathrm{tr}(\rho P) \right)^2 d\mu$$

$$\int \sum_{i,j,k,l} \langle E_j | P | E_i \rangle \left(a_i a_j^* - \frac{\delta_{ij}}{w} \right) \langle E_l | P | E_k \rangle \left(a_k a_l^* - \frac{\delta_{kl}}{w} \right) d\mu$$

The integral forces $i=j$ and $k=l$ or $i=l$, $k=j$

$$\int a_i a_j^* a_k a_l^* \, d\mu = \left(\delta_{ij} \delta_{kl} + \delta_{il} \delta_{jk} \right) \frac{1}{w(w+1)}$$

You can pesuade yourself that the integral is bounded above by the $\delta_{il} \delta_{kj}$ term

So we get

$$\delta^2 \leq \sum_{i,j} \frac{1}{w(w+1)} \langle E_i | P | E_j \rangle \langle E_j | P | E_i \rangle$$

But

$$\sum_{i,j} \langle E_i | P | E_j \rangle \langle E_j | P | E_i \rangle \leq \sum_i \langle E_i | P^2 | E_i \rangle$$

$\sum |E_j\rangle\langle E_j|$ is not complete in the full H-space.

$$= \sum \langle E_i | P | E_i \rangle \leq W$$

So $\langle \delta^2 \rangle \leq \frac{1}{W+1}$

This is a significant result!

Recall that $W = e^{S}$

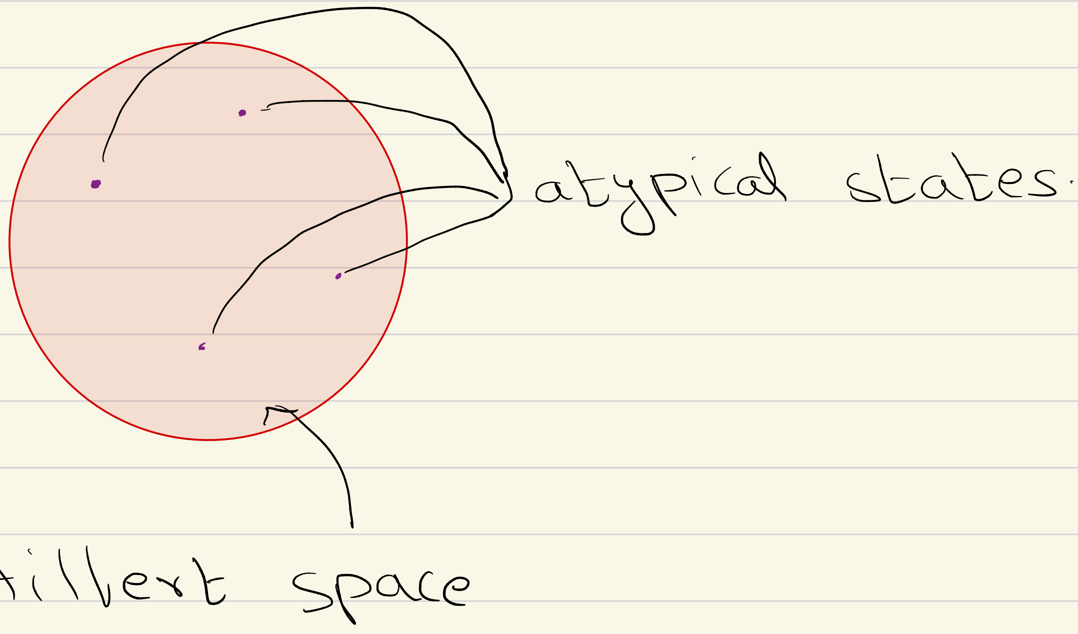
and $\frac{1}{\sqrt{W}} = e^{-S/2}$

Not only is it that pure states, on average, look like the microcanonical state the average deviation of such states from the mixed state is exponentially suppressed.

This result tells us:

a) For any observable, most states look almost like the mixed state

b) Volume of atypical states is exponentially small



This is quite unlike our intuition for smaller dimensional spaces.

In a high dimensional space, most states are "featureless".

Possible confusion: basis vs typical states

Sometimes, people give examples of qubits.

If we have n-qubits

$|000 \ldots 0\rangle$

$|100 \ldots 0\rangle$

$|1110.0 \ldots 0\rangle$

are all pure states and are special from the pov of σ_3^i

But basis states take up 0-volume in the H-space.

Possible confusion: no requirement for ETH

This is a Kinematical result

It does not require the eigenstate thermalization hypothesis or any assumption about the dynamics of the system.

Entanglement Entropy

The E.E. can distinguish pure and mixed states

But the E.E. is not the expectation value of any observable [proof in lecture]

It is a "function of correlation functions" that is sensitive to exponentially small effects.

Application to Hawking's original paradox.

The application to Hawking's original paradox is obvious.

We computed

$$\langle a_{\omega,l} \, a^{\dagger}_{\omega,l} \rangle = \frac{1}{1-e^{-\beta\omega}}$$

and

$$\langle a_{\omega,l} \, a^{\dagger}_{\omega',l} \rangle = 0 \quad \text{for} \quad \omega \neq \omega'$$

If these equations had been exact we could have concluded that the final state was mixed.

But if we just add $O(e^{-S/2})$ corrections

$$\langle a_{\omega,l}\, a^{\dagger}_{\omega,l} \rangle = \frac{1}{1-e^{-\beta\omega}} + O(e^{-S/2})$$

$$\langle a_{\omega,l}\, a^{\dagger}_{\omega',l} \rangle = O(e^{-S/2})$$

the results are perfectly consistent with a pure state.

In our computation (and in Hawking's) the accuracy was **far lower** than $O(e^{-S/2})$

For instance: say the b.h. was prepared in a microcanonical ensemble rather than canonical ensemble.

Then we expect $\frac{1}{\sqrt{S}}$ corrections to thermal correlators.

Even these corrections have not been considered systematically

Hawking's computation is not precise enough to lead to a paradox.