10 Feb 2020

Lecture 9: The Monogamy Paradox

We are now in a position to frame our second significant paradox about black hole evaporation.

The idea is that

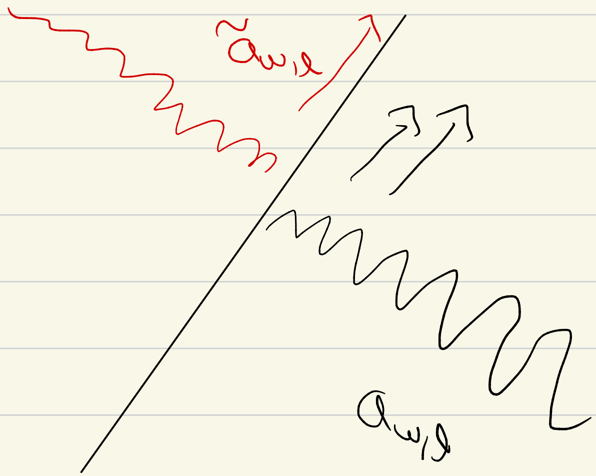
a) Modes across the horizon are entangled

b) Near horizon modes are entangled with far away radiation for an old black hole

c) This contradicts the monogamy of entanglement

Entanglement across the horizon

Consider again the modes near the horizon defined by $a_{\omega,l}$ and $\tilde{a}_{\omega,l}$



Both these modes are supported only near the horizon.
They are right movers on opposite sides of the horizon

Remember that these modes satisfy

$$\left(a_{\omega,l} - e^{-\beta\omega/2}\, \tilde{a}^{+}_{\omega,l}\right) |\psi\rangle = 0$$

$$\left(a^{+}_{\omega,l} - e^{\beta\omega/2}\, \tilde{a}_{\omega,l}\right) |\psi\rangle = 0$$

where $|\psi\rangle$ is the state of the full theory

In particular note that

$$\langle \psi | a_{\omega,l}\, \tilde{a}_{\omega,l} |\psi\rangle = \frac{e^{-\beta\omega/2}}{1-e^{-\beta\omega}}$$

We want to recast these correlators as CHSH correlators

The key point is that given the operator

$$N_{\omega,l} = a^{+}_{\omega,l} \, a_{\omega,l}$$

we can define a projector onto states with a definite value of $N_{\omega,l}$

Since the eigenvalues of $N_{\omega,l}$ are quantized this can be written as

$$|n\rangle\langle n| = \int e^{i\theta(N_{\omega,l}-n)} \frac{d\theta}{2\pi}$$

only notation

this is precise def^{n}.

Note that in the interacting theory perhaps $N_{\omega,l}$ will not have integrally quantized values

Nevertheless, this expression makes sense and gives an approximate projector.

We can also define

$$|n\rangle\langle n+1| \equiv \frac{1}{\sqrt{n+1}} |n\rangle\langle n| a_{\omega,l}$$

$$|n+1\rangle\langle n| \equiv \frac{1}{\sqrt{n+1}} a^\dagger_{\omega,l} |n\rangle\langle n|$$

Note the expressions on the RHS have concrete expansions as fields.

Similary, one can define

$$|\tilde{n}\rangle\langle\tilde{n}| \quad , \quad |\tilde{n}\rangle\langle\tilde{n}+1| \quad , \quad |\tilde{n}+1\rangle\langle\tilde{n}|.$$

eg

$$|\tilde{n}\rangle\langle\tilde{n}| = \int e^{i\theta(\tilde{N}_{\omega,\ell}-n)} \frac{d\theta}{2\pi}$$

Now define

$$A_1 = \sum_{n=0}^{\infty} \left(|2n+1\rangle\langle 2n+1| - |2n\rangle\langle 2n| \right)$$

$$A_2 = \sum_{n=0}^{\infty} \left(|2n+1\rangle\langle 2n| + |2n\rangle\langle 2n+1| \right)$$

Once again, note these are well-defined EFT operators.

Notice that

$$A_1^2 = \sum_{n,m} \left(|2n+1\rangle\langle 2n+1| - |2n\rangle\langle 2n| \right)\left(|2m+1\rangle\langle 2m+1| - |2m\rangle\langle 2m| \right)$$

We only need to use the fact that

$$|n\rangle\langle n| \quad |m\rangle\langle m| = \delta_{nm} \, |n\rangle\langle n|.$$

which can be proved in EFT

Then

$$A_1^2 = \sum_n |2n+1\rangle\langle 2n+1| + |2n\rangle\langle 2n|$$

$$= \sum_n |n\rangle\langle n| = 1.$$

Similarly
$$\tilde{A}_2^2 = \mathbb{1}$$

For future ref note that

$$A_1 A_2 = \sum_n \left(|2n+1\rangle\langle 2n| - |2n\rangle\langle 2n+1| \right)$$

and
$$A_2 A_1 = \sum_n |2n\rangle\langle 2n+1| - |2n+1\rangle\langle 2n|)$$

so
$$\{A_1, A_2\} = 0$$

and
$$(A_1 + A_2)^2 = 2$$

$\therefore$ A_1, A_2 and $\tilde{A}_1, \tilde{A}_2$ are "pseudospin" operators as we previously defined them.

Next, let us check

$$\langle \Psi | A_1 \tilde{A}_1 | \Psi \rangle .$$

These correlators can be computed directly but we can take a shortcut

Note

$$\tilde{N}_{\omega,\ell} |\Psi\rangle = \tilde{a}^\dagger_{\omega,\ell} \tilde{a}_{\omega,\ell} |\Psi\rangle = e^{-\beta\omega/2} \tilde{a}^\dagger_{\omega,\ell} a^\dagger_{\omega,\ell} |\Psi\rangle$$ ← using relation between modes

$$= a^\dagger_{\omega,\ell} \tilde{a}^\dagger_{\omega,\ell} |\Psi\rangle e^{-\beta\omega/2}$$ ← using commutator

$$= a^\dagger_{\omega,\ell} a_{\omega,\ell} |\Psi\rangle$$ ← relation between modes

$$= N_{\omega,\ell} |\Psi\rangle$$

We can prove similar relations for a power of $N_{\omega,l}$ also.

$$\tilde{N}_{\omega,l}^{a} |\psi\rangle = N_{\omega,l}^{a} |\psi\rangle$$

This tells us that projecting onto a given value of $N_{\omega,l}$ is the same as projecting onto $\tilde{N}_{\omega,l}$.

So we can **think** of the state as

$$|\psi\rangle \sim \sum a_n |n, \tilde{n}\rangle \otimes \psi_n^{rest}$$

and imposing $\left(\tilde{a}_{\omega,l} - e^{-\beta\omega/2} a_{\omega,l}^{\dagger}\right)|\psi\rangle = 0$

we can further simplify this to

$$|\psi\rangle = \left((1-e^{-\beta\omega})^{1/2} \sum e^{-\beta n\omega/2} \, |n, \tilde{n}\rangle \right) \otimes \psi_{rest}$$

Another way to say this is that **effectively** the density matrix looks like

$$\rho_{eff} = (1-e^{-\beta\omega}) \sum_n |n, \tilde{n}\rangle\langle n, \tilde{n}| \, e^{-\beta n\omega}.$$

This is **NOT** an exact statement or claim.

It only means that at **LEADING ORDER** one can use the density matrix above to compute correlators of simple operators and get the right answer.

Using this we immediately find that

$$\langle\psi| A_1 \tilde{A}_1 |\psi\rangle$$

$$= \langle\psi| \left(|2n+1\rangle\langle 2n+1| - |2n\rangle\langle 2n|\right)$$

$$\otimes \left(|2\tilde{n}+1\rangle\langle 2\tilde{n}+1| - |2\tilde{n}\rangle\langle 2\tilde{n}|\right) |\psi\rangle$$

Cross terms do not click because there is no component of $|\psi\rangle$ with $N = 2n+1$ and $\tilde{N} = 2n$ or vice-versa.
So again we get

$$\langle\psi| A_1 \tilde{A}_1 |\psi\rangle = 1.$$

Now consider

$$\langle\psi| A_2\tilde{A}_2 |\psi\rangle.$$

$$= \langle\psi| \left(|2n+1\rangle\langle 2n| + |2n\rangle\langle 2n+1|\right)$$

$$\otimes \left(|2\tilde{n}+1\rangle\langle 2\tilde{n}| + |2\tilde{n}\rangle\langle 2\tilde{n}+1|\right) |\psi\rangle$$

$$= \langle\psi| \left(|2n+1, 2\tilde{n}+1\rangle\langle 2n, 2\tilde{n}| + |2n, 2\tilde{n}\rangle\langle 2n+1, 2\tilde{n}+1|\right) |\psi\rangle$$

Lets invoke our density matrix to find this sum. We get

$$\sum_n 2 e^{-\beta(2n+1)\omega/2} e^{-\beta 2n\omega/2} (1 - e^{-\beta\omega})$$

$$= \frac{2e^{-\beta\omega/2}}{1+e^{-\beta\omega}}$$

We are almost done.

Define

$$B_1 = \cos\theta\, \tilde{A}_1 + \sin\theta\, \tilde{A}_2$$

$$B_2 = \cos\theta\, \tilde{A}_1 - \sin\theta\, \tilde{A}_2$$

Note that

$$B_1^2 = B_2^2 = 1 \quad \text{since} \quad \{\tilde{A}_1, \tilde{A}_2\} = 0$$

$$\text{and} \quad \tilde{A}_1^2 = \tilde{A}_2^2 = 1$$

Define

$$C_{AB} = \qquad A_1(B_1 + B_2) + A_2(B_1 - B_2)$$

Then

$$\langle \psi | C_{AB} | \psi \rangle = 2\cos\theta \langle \psi | A_1 \tilde{A}_1 | \psi \rangle$$
$$+ 2\sin\theta \langle \psi | A_2 \tilde{A}_2 | \psi \rangle$$
$$= 2\cos\theta + \frac{4 e^{-\beta\omega/2}}{1+e^{-\beta\omega}} \sin\theta$$

This can take a range of values but it is maximized when

$$\tan\theta = \frac{2 e^{-\beta\omega/2}}{1+e^{-\beta\omega}}$$

The max value is

$$\langle\psi| C_{AB} |\psi\rangle = \frac{2}{1+e^{-\beta\omega}} \left(1+6e^{-\beta\omega}+e^{-2\beta\omega}\right)^{1/2}$$

You can see this is always larger than 2 because

$$(1+e^{-\beta\omega})^2 = 1+e^{-2\beta\omega}+2e^{-\beta\omega}$$
$$< 1+6e^{-\beta\omega}+e^{-2\beta\omega}$$

For $e^{-\beta\omega}=1$, this is $2\sqrt{2}$.

Summary

a) The correlators across the horizon that we analyzed imply a violation of Bell inequalities.

$$\langle\psi|C_{AB}|\psi\rangle > 2$$

ii) Correlators are computed in EFT. Adding interactions, changing smearing ets. leads to

$$\delta\langle\psi|C_{AB}|\psi\rangle = \varepsilon$$

But since

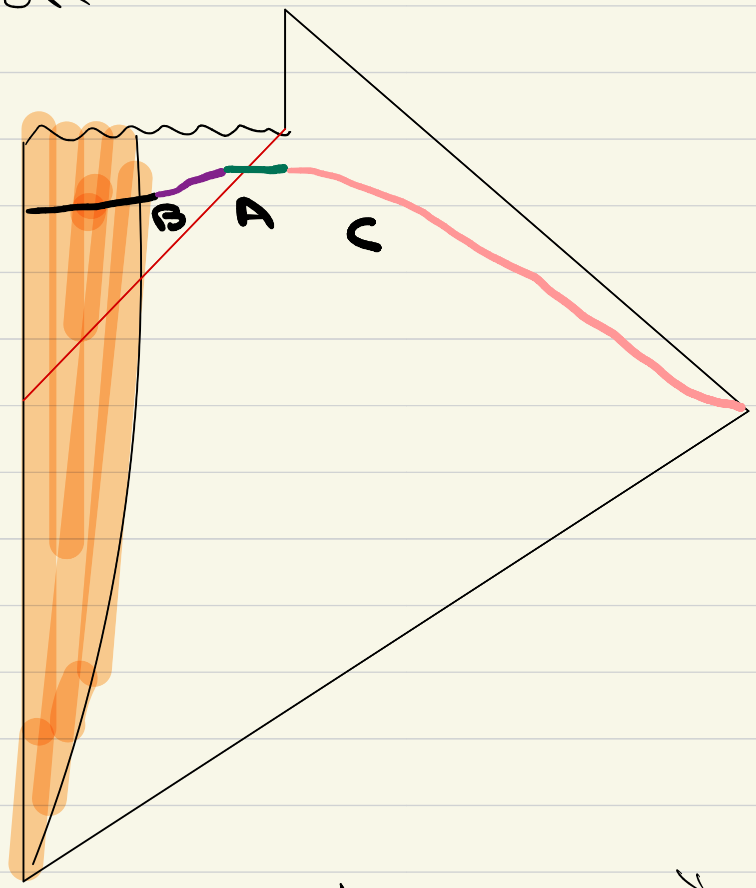
$$\langle\psi|C_{AB}|\psi\rangle - 2 = O(1)$$

We continue to have

$$\langle\psi|C_{AB}|\psi\rangle > 2.$$

So far the analysis has been robust.

Now let us turn to black hole evaporation



We want to draw a "nice slice" through the geometry at late times.

Let

B = region just behind horizon

A = region just outside horizon

C = region far away that contains the past Hawking radiation

Now we need to wave our hands and make assumptions!

1) First, the Hilbert space is infinite dimensional, but we assume that we can formulate dynamics effectively in an e^S dimensional Hilbert space. Lets call this "$\mathcal{H}$"

[Reasonable assumption]

2) We assume that the Hilbert space effectively **Factorizes** so that

$$\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \subset \mathcal{H}_{\text{``}}$$

This seems so reasonable that this assumption was not even stated explicitly in the Mathur / AMPS papers

3) The modes $a_{\omega,l}$, $a^{+}_{\omega,l}$ represent degrees of freedom in A, whereas the modes $\tilde{a}_{\omega,l}$ represent dof in B

4) Early Hawking radiation is captured by "C".

This may appear "sloppy" because it is sloppy!

Now consider an **OLD** black hole which has crossed the half-way point of evaporation

so

Complement of $\mathcal{H}_c$

$$\dim(\mathcal{H}_c) \gg e^{S/2} \gg \dim(\overline{\mathcal{H}}_c)$$

Now note that we have the operators A_1, A_2 which act on $\overline{\mathcal{H}}_c$

So by the result from the last lecture

we can find operators C_1, C_2 acting within H_C so that

$$\langle \psi | C_{AC} | \psi \rangle = 2\sqrt{2}$$

up to small corrections!

But now we have a paradox since we earlier found that

$$\langle \psi | C_{AB} | \psi \rangle > 2$$

So

$$\langle \psi | C_{AB} | \psi \rangle^2 + \langle \psi | C_{AC} | \psi \rangle^2 > 8!$$

Note a few salient points

1) This paradox makes reference to the interior since it makes ref to C_{AB}

2) The violation is an $O(1)$ violation which looks difficult to fix using small corrections.