QABH - Lecture 15, 14 Oct 2016

We will now put our analysis of black holes and curved-space QFT together to derive the formula for the temperature of a black hole.

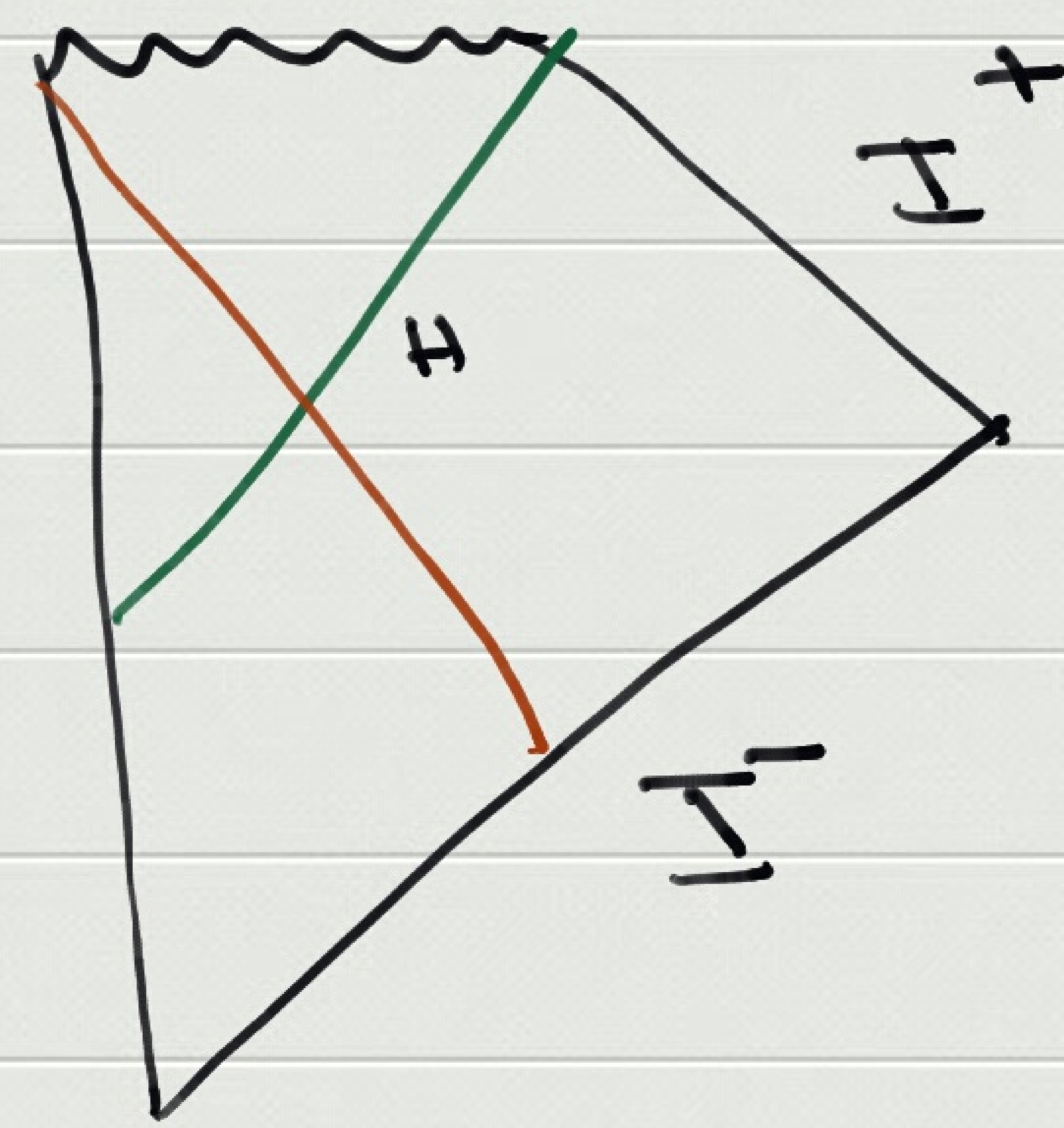
Previously we proved

$$dM = \frac{\kappa}{8\pi}\, dA + \Omega\, dJ$$

Today we will argue that $\frac{\kappa}{2\pi}$ is the temperature.

We will consider Hawking's original derivation using ray tracing. We will discuss problems with this derivation and then turn to more precise derivations later

Consider a B.H. formed from collapse. For simplicity we consider a single shell collapsing at the speed of light.



We can specify initial data for the fields at past null infinity $\mathcal{I}^-$ or on $H \cup \mathcal{I}^+$

So we have two possible expansions

$$\phi = \sum a_i f_i(r,t,\Omega) + h.c.$$

specified by demanding that at past null infinity, this becomes

$$\phi = \sum_m \int \frac{a_\omega}{\sqrt{\omega}} \frac{e^{-i\omega v}}{r} Y_m(\Omega) + h.c.$$

This form comes a spherical wave expansion and then by taking $r \to \infty$ $t \to -\infty$ but $t+r$ finite

The second expansion is

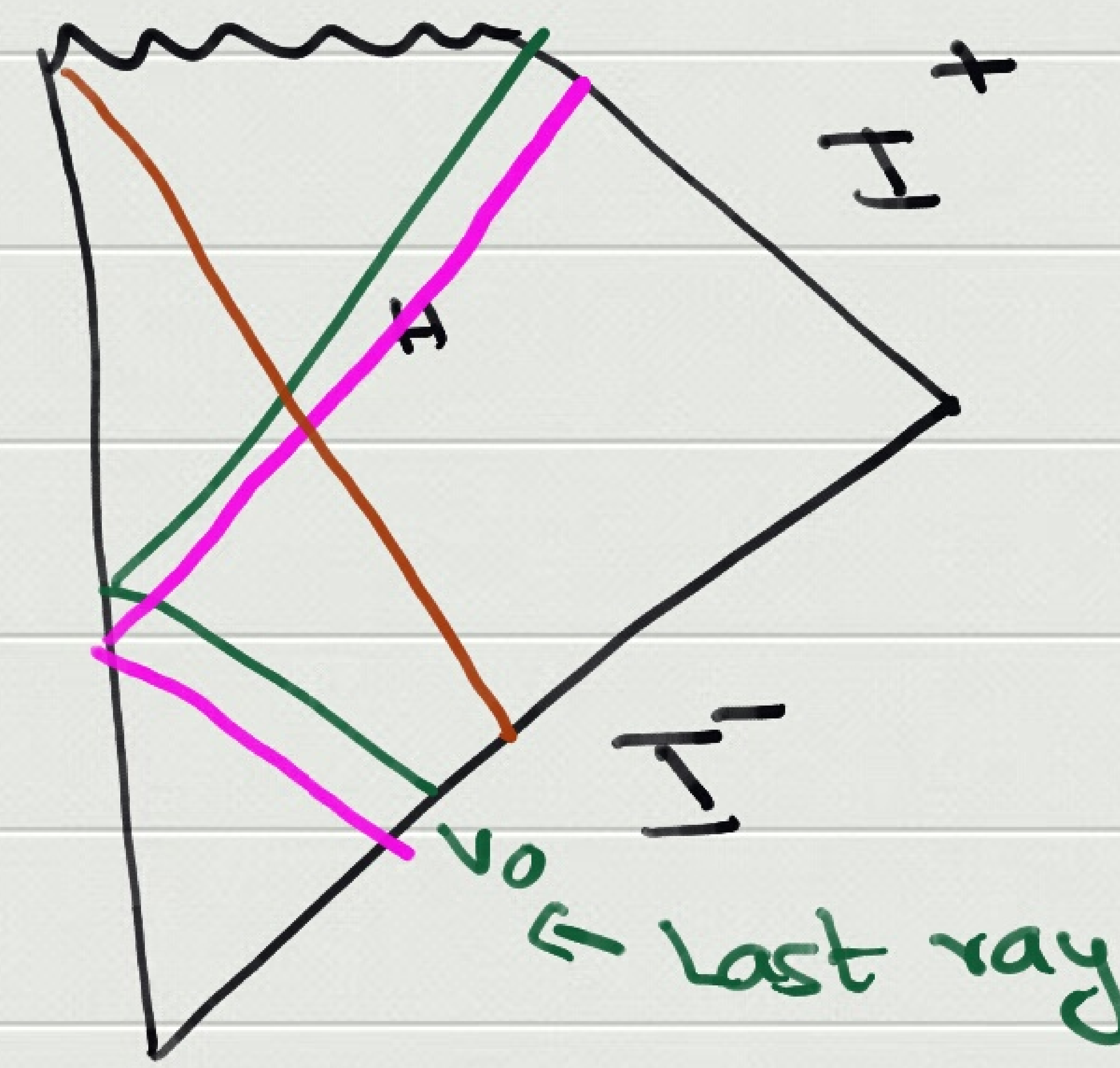
$$\phi = \sum b_i g_i(r,t,\Omega) + h.c. + c_i h_i(r,t,\Omega) + h.c.$$

where the g_i have support at future null infinity so that there

$$\phi \to \sum_m \int \frac{b_\omega}{\sqrt{\omega}} \frac{e^{-i\omega u}}{r} Y_m(\Omega) + h.c.$$

Both the c modes and the m index does not have much of a role to play in what follows.

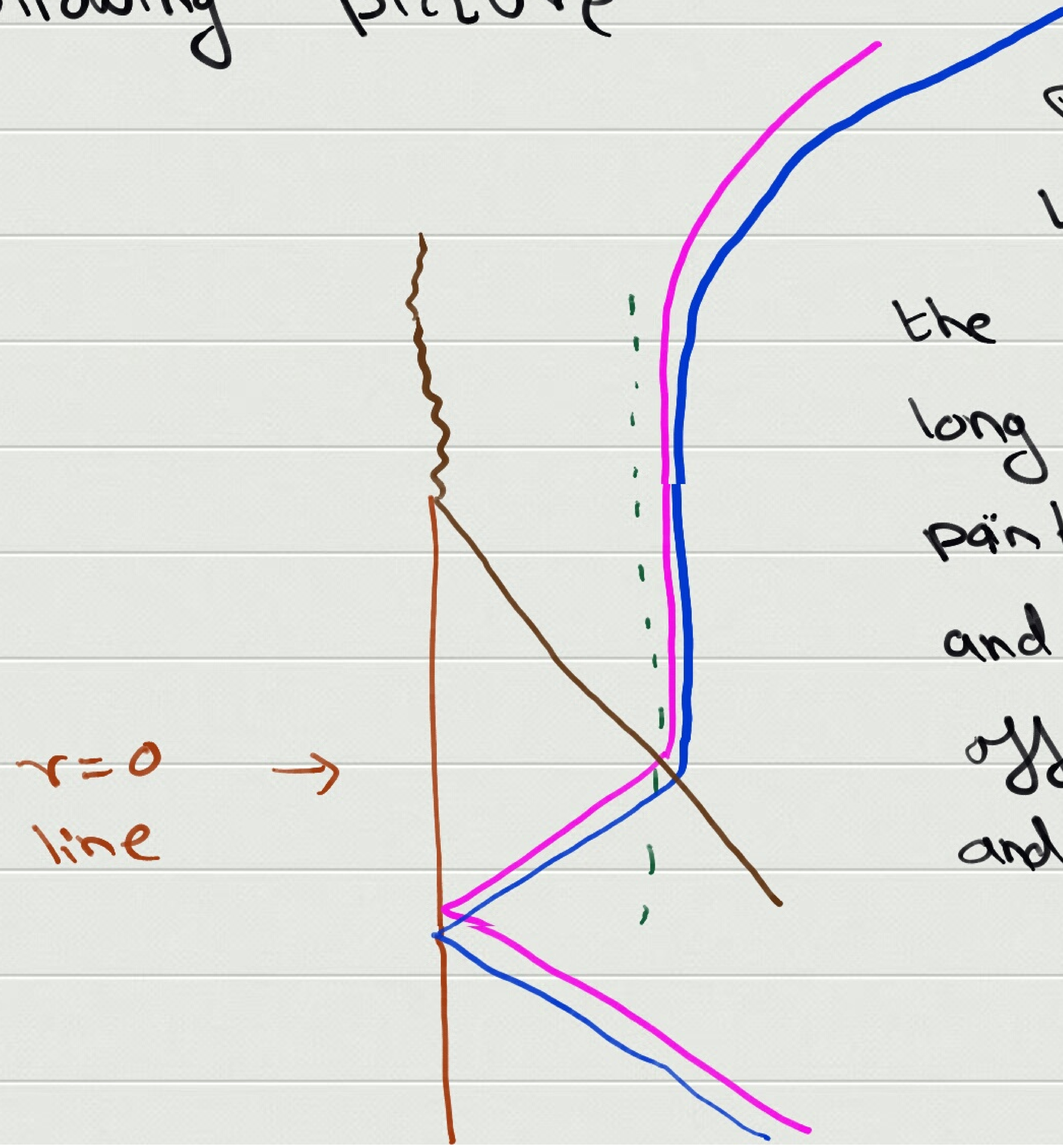
Consider the Penrose diagram again



We see modes on $\mathcal{I}^+$ can be related to modes with support just befor the last ray at v_0 using

RAY TRACING

To make this more physical, we can view this in the following picture



If we trace from future to past, Light rays get stuck at the event horizon for a long time. Then at some point they enter the shell and propagate quickly, bounce off the origin of polar coordinates and back out to infinity.

Outside the shell, the trajectory of the light ray is along

$$\delta r_* = \delta t$$

The key point is that after it enters the shell, we have

$$\delta r = \delta \tau$$

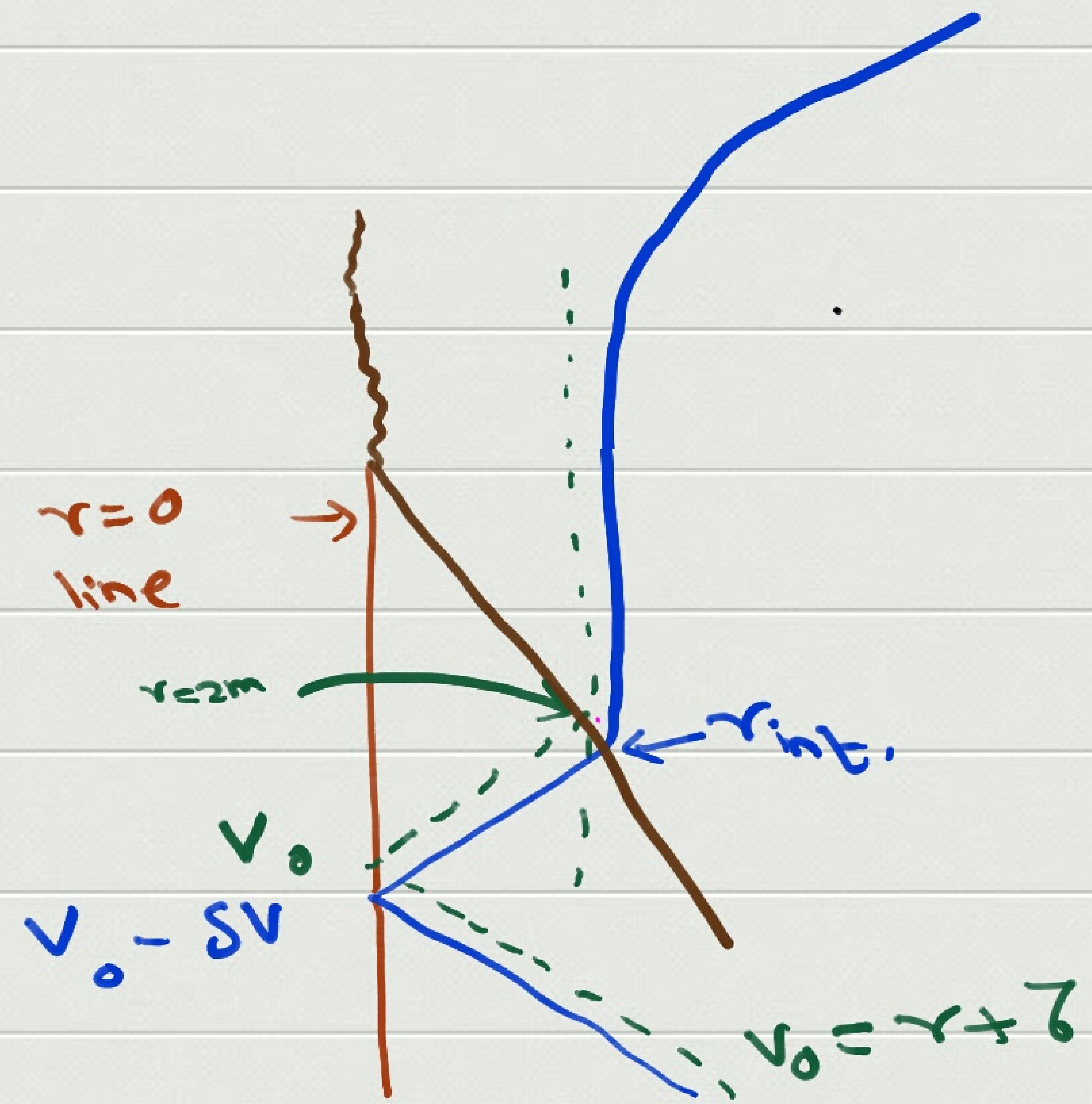
defined using radius of S^2

because now we are in empty space.

When it bounces off $r=0$, it is still in empty space and so, again

$$|\delta r| = |\delta t|.$$

Now consider two rays. One is the "last ray" that gets stuck at the event horizon. The second is a ray slightly outside



For the purposes of doing geometric analysis, it is useful to introduce two coordinate systems.

$$\left.\begin{aligned} V &= t + r \\ U &= r - t \end{aligned}\right\} \text{ inside the shell}$$

$$\left.\begin{aligned} V^{out} &= t + r_* \\ U^{out} &= r_* - t \end{aligned}\right\} \text{ outside the shell}$$

We can choose the origin of time so that the shell is at $V = V_0$ in both coordinates.

r is the radius of the sphere and also continuous.

If the last ray starts from $\mathcal{I}^-$ at v_0 and the other ray starts at $v_0 - \delta v$, then the last reaches the origin of polar coordinates at v_0 and the other one at $v_0 - \delta v$

The last ray bounces off the origin on a trajectory

$$u = -v_0 = r - t$$

the shell is coming in at

$$v = v_s = r + t$$

these two intersect at $r = 2m \Rightarrow v_s = 4m + v_0$

The earlier ray is on a trajectory (after bouncing) of

$$U_{ray} = -V_0 + \delta V$$

So it intersects the shell at

$$r_{int} = \frac{V_S - V_0 + \delta V}{2} = 2m + \frac{\delta V}{2}$$

The corresponding value of

$$r_* = r_{int} + 2m \ln \left| \frac{r_{int} - 2m}{2m} \right|$$

$$= 2m + \frac{\delta V}{2} + 2m \ln \left| \frac{\delta V}{4m} \right|$$

Recall that even in the outside coordinates, the shell is at $V^{out} = V_s$.

So

$$\boxed{\begin{aligned} U^{out} &= 2r_* - V_s^{out} \\ &= 4m + \delta V + 4m \ln \frac{\delta V}{4m} - V_s. \end{aligned}}$$

$\uparrow$

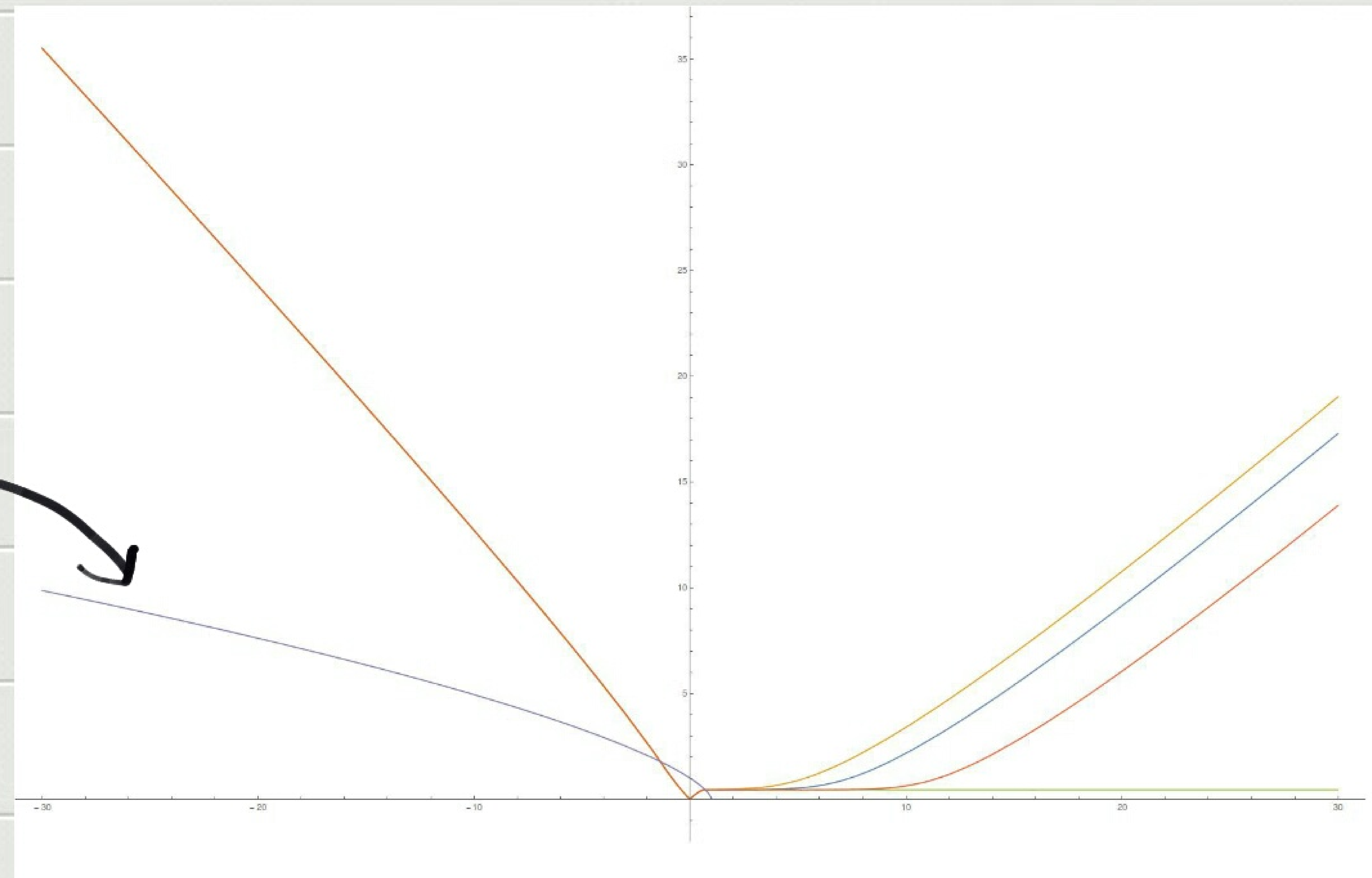
central ray tracing formula

Now in deriving this formula we made some specific assumptions. In particular that the incoming shell was thin and collapsing at the speed of light.

If we change these assumptions, some specific details in the formula chang.

eg. recall the Oppenheimer-Snyder geometry

We considered geodesics in this background



$\uparrow x$ = radius of sphere

surface of star

$\rightarrow \tau$

For some choice of parameters the geodesics look as above

Once again we can divide spacetime into the region inside the star and outside the star. once again inside the star geodesic propagation is well approximated by $\delta r = \pm \delta t$

and once again if the last ray intersects the star's outer surface at $2m$, the earlier ray intersects at $r_{int} = 2m + \alpha\, \delta v$

$\uparrow$
depends on details of collapse

If the intersection of the last ray with matter is at $v = v_s$

then the earlier ray has a U^{out} coord given by

$$U^{out} = 2r_*^{int} - V_s = r + \beta\,\delta V + 4m \ln \frac{\delta V}{\alpha}$$

depend on details.

But very close to the last ray the relation

$$U^{out} \sim 4m \ln \delta V$$

is robust.

Now the argument about ray tracing is as follows.
Note that an infinite region (in affine coordinates) on $\mathscr{I}^+$
is filled by rays from a small region near v_0 on $\mathscr{I}^-$

So rays at $O(1)$ frequencies on $\mathscr{I}^+$ are blueshifted when we trace them back to $\mathscr{I}^-$. For high-freq waves (and for free propagation), ray tracing gives a good approximation to wave propagation
i.e. rays approximate a curve on which phase is constant and phase-difference between neighbouring rays is preserved.

This suggests that if we take a solution

$$e^{i\omega U} \leftarrow \text{on } \mathscr{I}^+$$

it propagates to the solution

$$\left(\frac{V_0 - V}{4m}\right)^{4im\omega} \qquad \text{with } V < V_0$$

on $\mathscr{I}^-$

[Using $U \sim 4m \ln \frac{\delta V}{4m}$, for small $\frac{\delta V}{4m}$]

Note this does not give a complete basis on $\mathscr{I}^-$. For that we need the solutions

$$(V - V_0)^{\omega i m \omega} \quad , \quad \text{with } V > V_0$$

These correspond to the c-modes.

For now let us focus on the b-modes

Then the mathematical problem we have is as follows.

In the past, on $\mathscr{I}^-$ we consider the expansion

$$\phi = \int \frac{a_\omega}{\sqrt{\omega}} \frac{e^{ikv}}{r} Y_{lm}(\Omega) + h.c.$$

and put the universe in the state

$$a_\omega |\Omega\rangle = 0$$

At late times, we consider

$$\Phi = \int \frac{b_\omega}{\sqrt{\omega}} \left(\frac{V_0 - V}{r}\right)^{4im\omega} Y_m(\Omega) + h.c. \quad , \quad V < V_0$$

$$= \int \frac{c_\omega}{\sqrt{\omega}} \left(\frac{V - V_0}{r}\right)^{4im\omega} Y_m(\Omega) + h.c. \quad V > V_0$$

We want the final state in the l-c vacuum.

But this is just the problem of the Bogoliubov coefficients in Rindler space.

The answer is the state $|\Omega\rangle$ appears like a thermal density matrix for the observer at $\mathscr{I}^+$ [who observes only b_ω] and the temperature is

$$\beta = 8\pi M .$$