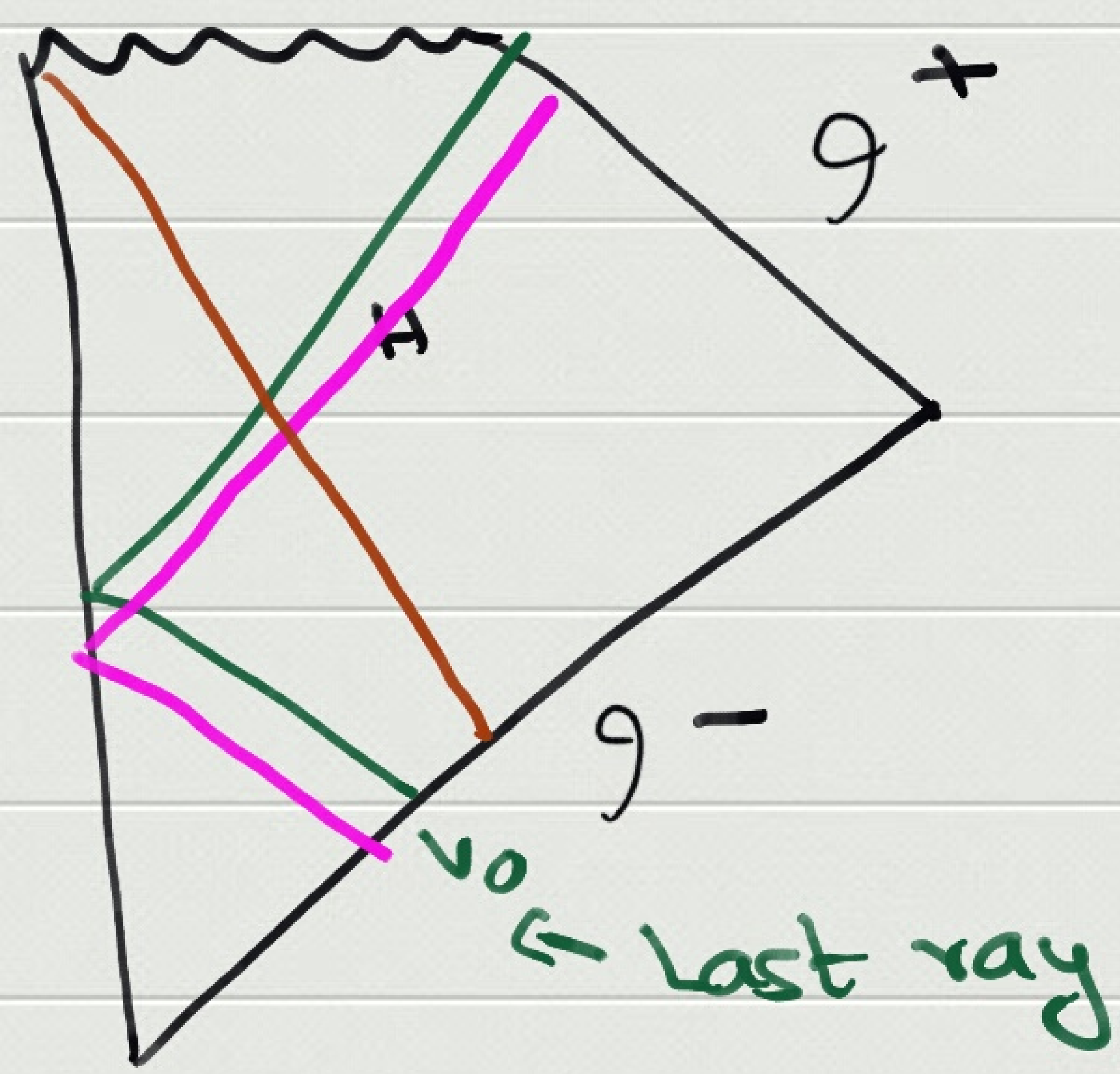
Quantum Aspects of Black Holes - Lecture 16

Last time we derived the formula for the Hawking temperature through ray tracing.



We related the position of a ray on $\mathcal{I}^-$ to its final position on $\mathcal{I}^+$

$$U = \alpha + \beta \delta v + 4m \ln \delta v$$

α and β depended on the details of the collapse, but we argued the log term was universal.

What this means is that, in the geometric optics approximation, if the wave on $\mathcal{I}^-$ can be written as

$$\phi \sim e^{i\psi_-(v)}$$

and the same solution tends to

$$\phi \sim e^{i\psi_+(u)}$$

only phase diff is preserved during ray tracing

then

$$\psi_+(U = \alpha + \beta \delta V + 4m \ln \delta V) = \psi_-(V) + \text{constant}$$

Say we consider a typical frequency on $\mathscr{I}^+$.
This means the frequency

$$\omega_+ \sim O\left(\frac{1}{m}\right) = O(T)$$

then we want

$$\frac{d\psi_+}{dU} = \omega_+$$

but then $$\frac{d\psi_-}{dV} = \omega_+ \frac{dU}{dV} = \omega_+\left(\beta + \frac{4m}{\delta V}\right)$$

So as we take U to be large, δV becomes small and $\omega_- \rightarrow \infty$!

This is called the trans-Planckian problem. The Hawking radiation at late-times comes from ultra-blueshifted frequencies in the past.

This seems to go against effective field theory.

Physics at late times depends on the deep ultraviolet

What if we put a cutoff and truncate the

spectrum so that no frequencies above some Λ occur?
will Hawking radiation stop after some v?

[explain this more in class]

So we look for a more precise derivation at late times.

The first step is to solve the wave-eqn.
we write

$$\phi = R(r_*)\, Y_{l,m}(\theta,\phi)\, e^{-i\omega t}.$$

Recall the metric is

$$ds^2 = \left(1-\frac{2m}{r}\right)\left(dt^2 - dr_*^2\right) - r^2 d\Omega^2$$

so $$\sqrt{-g} = \left(1-\frac{2m}{r}\right) r^2 \sin\theta$$

the wave-eqn is

$$\frac{1}{\sqrt{-g}}\, \partial_\mu\, g^{\mu\nu} \sqrt{-g}\, \partial_\nu\, \phi = 0$$

or

$$\frac{1}{\left(1-\frac{2m}{r}\right)} \partial_t^2 \phi - \frac{\partial_{r_*} r^2 \partial_{r_*} \phi}{r^2\left(1-\frac{2m}{r}\right)} - \frac{1}{r^2} \Box_{\Omega_2} \phi = 0$$

which leads to

$$\frac{1}{r^2} \partial_{r_*} r^2 \partial_{r_*} \phi + \omega^2 \phi - \left(1-\frac{2m}{r}\right) \frac{l(l+1)}{r^2} = 0$$

This cannot be solved in terms of elementary functions. The standard procedure is to use the WKB approximation in various limits.

We will only consider two limits

$r \rightarrow 2m$

and $r \rightarrow \infty$

In both these limits the equation simplifies.

recall $\frac{dr}{dr_*} = \left(1 - \frac{2m}{r}\right)$

as $r \to 2m$, we can write

$$\phi = A e^{i\omega r_*} + B e^{-i\omega r_*}$$

as $r \to \infty$, we can write

$$\phi = \frac{C}{r} e^{i\omega r_*} + \frac{D}{r} e^{-i\omega r_*}$$

of course only two of these four solutions are independent.

In the ray-tracing derivation, we looked for solutions with support on $\mathscr{I}^+$ and H.

This corresponds to solutions ingoing at the horizon and a second solution outgoing at infinity, i.e.

we consider

outgoing at ∞ $\longrightarrow$ $f_{\omega,l} \xrightarrow[r\to\infty]{} \dfrac{A}{r} e^{i\omega r_*}$

$\xrightarrow[r\to 2M]{} \dfrac{1}{r} e^{i\omega r_*} + \dfrac{B e^{-i\omega r_*}}{r}$

ingoing at hor $\longrightarrow$ $g_{\omega,l} \xrightarrow[r\to\infty]{} \dfrac{C}{r} e^{i\omega r_*} + \dfrac{1}{r} e^{-i\omega r_*}$

$\xrightarrow[r\to 2M]{} D \dfrac{e^{-i\omega r_*}}{r}.$

The normalization of the solution is fixed by the c.c. relations,

$$\left[\phi(t,r,\Omega)\,,\,\sqrt{-g}\,g^{00}\,\dot{\phi}(t,r',\Omega')\right] = i\,\delta(r-r')\,\delta(\Omega-\Omega').$$

The constants A, B, C, D need to be calculated using the wave equation.

So we expand the field as

$$\phi = \sum_{\omega,l} \frac{1}{\sqrt{\omega}} \Big[a_{\omega,l}\, f_{\omega,l}(r)\, e^{-i\omega t}\, Y_{l,m}(\theta,\phi)$$

$$+\, b_{\omega,l}\, g_{\omega,l}(r)\, e^{-i\omega t}\, Y_{l,m}(\theta,\phi) + h.c. \Big]$$

The result then is as follows. If we impose that there is no incoming radiation from infinity then this sets

$$b_\omega |U\rangle = 0.$$

Furthermore regularity of the future horizon requires

$$\langle U| a^\dagger_\omega a_{\omega'}|U\rangle = \frac{e^{-\beta\omega}}{1-e^{-\beta\omega}} \delta(\omega-\omega').$$

The state $|U\rangle$ is called the Unruh vacuum.

Our derivation is a version of one of the derivations we used in the Rindler/Minkowski Bogoliubov transformations.

We consider the two-point function

$$\langle \phi(t, r, \Omega) \, \phi(t', r', \Omega') \rangle$$

in the near-horizon limit. Geometry regular $\Rightarrow$ this correlator is regular.

We now show that correlator regular $\Rightarrow$ $\langle U | b_\omega^\dagger b_{\omega'} | U \rangle$ is thermal.

For this derivation it is convenient to go to Kruskal coordinates defined through

$$U = - e^{\frac{2\pi}{\beta}(r_* - t)}$$

$$V = e^{\frac{2\pi}{\beta}(t + r_*)}.$$

then the near horizon metric is given by

$$ds^2 = -K\, dU\, dV + r_h^2\, d\Omega^2$$

So if we take points to be very close we must have

$$\langle \phi(U_1, V_1, \Omega_1)\, \phi(U_2, V_2, \Omega_2) \rangle = \frac{1}{-k\, \delta U\, \delta V + r_h^2\, \delta\Omega^2}$$

This is necessary for smoothness of the near-horizon region. A corollary is:

$$\lim_{\delta V \to 0} \langle \partial_{U_1} \phi(U_1, V_1, \Omega_1)\, \partial_{U_2} \phi(U_2, V_2, \Omega_2) \rangle \propto \frac{\delta^2(\delta\Omega)}{(\delta U)^2}$$

Now consider the near-horizon expansion of the field

$$f_{\omega,\ell}(r_*)\, e^{-i\omega t} \xrightarrow[r \to 2M]{} \frac{1}{2m}(-U)^{i\beta\omega/2\pi} + \frac{B}{2m} V^{-i\beta\omega/2\pi}$$

$$g_{\omega,\ell}(r_*)\, e^{-i\omega t} \xrightarrow[r \to 2M]{} \frac{D}{2m} V^{-i\beta\omega/2\pi}$$

so

normalization const.

$$\langle \partial_{U_1}\phi_1\, \partial_{U_2}\phi_2 \rangle = N \int d\omega\, \omega \cdot \frac{1}{U_1 U_2}\left[\langle a_\omega^\dagger a_\omega \rangle \left(\frac{U_2}{U_1}\right)^{i\beta\omega/2\pi} + \langle a_\omega a_\omega^\dagger \rangle \left(\frac{U_1}{U_2}\right)^{i\beta\omega/2\pi} \right]$$

ω^2 from derivative

cancels $\frac{1}{\omega}$ from fields

$$\delta^2(\delta\Omega)$$

comes from angular integrals

If we now set

$$\langle a_\omega^+ a_\omega \rangle = \frac{e^{-\beta\omega}}{1-e^{-\beta\omega}}$$

$$\langle a_\omega a_\omega^+ \rangle = \frac{1}{1-e^{-\beta\omega}}$$

the integral becomes

$$\langle \partial_{U_1}\phi_1 \; \partial_{U_2}\phi_2 \rangle = N \frac{\delta^2(\delta\Omega)}{U_1 U_2} \int d\omega \; \omega \left(\frac{U_1}{U_2}\right)^{i\beta\omega/2\pi} \frac{1}{1-e^{-\beta\omega}} \, d\omega$$

For $|v_1| > |v_2|$ we can complete the contour in the upper-half plane and get

$$\langle \partial_{v_1}\phi_1 \, \partial_{v_2}\phi_2 \rangle \propto \frac{\delta^2(\delta\Omega)}{v_1 v_2} \frac{v_2/v_1}{\left(1-\frac{v_2}{v_1}\right)^2}$$

$$= \frac{\delta^2(\delta\Omega)}{(v_1-v_2)^2}$$

exactly as we needed!

Note that $\langle a_\omega^\dagger a_\omega \rangle = \dfrac{e^{-\beta\omega}}{1-e^{-\beta\omega}}$ does not imply an exactly thermal spectrum at $\mathscr{I}^+$. This is because the coefficient A in our defn of f [see page 10] is not identically 1.

Some of the emitted radiation is back-scattered by the potential and does not make it out. This is the physics of greybody factors.