Quantum Aspects of Black Holes - Lecture 17 - 21 Oct 2016.

Last time we derived the formula for Hawking radiation by demanding locally that the future horizon was smooth.

Today, we will discuss entanglement across the horizon.

Recall that in the ray-tracing derivation, rays before the last ray were correlated with rays after the last ray behind the horizon.

To see a more precise version of this, we need to quantize the field behind the horizon.

We currently have:

$$\phi = \sum_{\omega,l} \frac{1}{\sqrt{\omega}} \Big[a_{\omega,l}\, f_{\omega,l}(r)\, e^{-i\omega t}\, Y_{l,m}(\theta,\phi)$$

$$+\, b_{\omega,l}\, g_{\omega,l}(r)\, e^{-i\omega t}\, Y_{l,m}(\theta,\phi) + h.c. \Big]$$

where

outgoing at ∞ $\longrightarrow$ $f_{\omega,l} \xrightarrow[r\to\infty]{} \frac{A}{r} e^{i\omega r_*}$

$\xrightarrow[r\to 2M]{} \frac{1}{r} e^{i\omega r_*} + \frac{B e^{-i\omega r_*}}{r}$

ingoing at hor $\longrightarrow$ $g_{\omega,l} \xrightarrow[r\to\infty]{} \frac{C}{r} e^{i\omega r_*} + \frac{1}{r} e^{-i\omega r_*}$

$\xrightarrow[r\to 2M]{} D\frac{e^{-i\omega r_*}}{r}.$

We need fresh right-moving modes behind the horizon. and they must be correctly entangled with modes outside if we wish to preserve a smooth horizon and local commutators.

First note that the terms proportional to

$$e^{-i\omega r_*} \quad e^{-i\omega t}$$

cross smoothly across the horizon.

But recall the c.c. relation inside the horizon require

$$[\phi(r_*, t, \Omega), \frac{\partial \phi}{\partial r_*}(r_*, t', \Omega')] = \frac{1}{\sin\theta}\delta(\Omega-\Omega')\delta(t-t')$$

Now, r_* is the time-coordinate!

We have not been so careful with factors of 2. But note that if our expansion only has terms [near-horizon]

$$\phi = \sum_{\omega > 0} \frac{c_\omega}{\sqrt{\omega}} e^{-i\omega(r_*+t)} + h.c. \quad ; \quad c_\omega = B a_\omega + D b_\omega$$

and c_ω's are normalized to satisfy the c.c. relation outside the horizon (where there is an additional degree of freedom, then they cannot alone satisfy c.c. relations inside.

So we require additional modes behind the horizon.

we can write

$$\phi = \sum \frac{1}{\sqrt{\omega}} \frac{g_{\omega,\ell}}{D_{\omega,\ell}}(r_*) Y_{\ell,m}(\Omega)\left(c_\omega e^{-i\omega t} + d_\omega e^{i\omega t}\right) + h.c.$$

where

$$\frac{g_{\omega,\ell}}{D_{\omega,\ell}} \xrightarrow[r_* \to -\infty]{} e^{-i\omega r_*}$$

Note that even though d_ω multiplies $e^{i\omega t}$ it is an annihilation operator.

Now we can repeat the derivation from last time to work out

$$\langle d_\omega \, a_\omega \rangle$$

and $\langle d_\omega^\dagger \, a_\omega^\dagger \rangle$

Once again we consider the two-point function but now across the horizon.

This is very similar to our previous derivation

Define Kruskal coordinates defined through

$$U = - e^{\frac{2\pi}{\beta}(r_* - t)} \quad , \text{ outside}$$

$$V = e^{\frac{2\pi}{\beta}(t + r_*)}.$$

But

$$U = e^{2\pi/\beta\,(r_* - t)} \quad , \text{ inside}$$

then the near horizon metric is given by

$$ds^2 = -K\, dU\, dV + r_h^2\, d\Omega^2$$

So if we take points to be very close we must have

$$\langle \Phi(U_1, V_1, \Omega_1)\, \Phi(U_2, V_2, \Omega_2) \rangle = \frac{1}{-k\, \delta U\, \delta V + r_h^2\, \delta\Omega^2}$$

[Now we have $U_1 < 0$, $U_2 > 0$; but the relation above is still true]

This is necessary for smoothness of the near-horizon region. A corollary is:

$$\lim_{\delta V \to 0} \langle \partial_{U_1} \Phi(U_1, V_1, \Omega_1)\, \partial_{U_2} \Phi(U_2, V_2, \Omega_2) \rangle \propto \frac{\delta^2(\delta\Omega)}{(\delta U)^2}$$

Now consider the near-horizon expansion of the field

$$f_{\omega,\ell}(r_*)\,e^{-i\omega t} \xrightarrow[r\to 2M]{} \frac{}{2m}(-U)^{i\beta\omega/2\pi} + \frac{B}{2m} V^{-i\beta\omega/2\pi}$$

$$\frac{g_{\omega,\ell}}{D_{\omega,\ell}}(r_*)\,e^{i\omega t} \xrightarrow[r\to 2M]{} \frac{}{2m} U^{-i\beta\omega/2\pi}$$

Not absence of minus sign

So

normalization const.

$$\langle \partial_{U_1}\phi_1\ \partial_{U_2}\phi_2\rangle = N\int d\omega\,\omega\ \frac{1}{U_1U_2}\left[\langle a_\omega a_\omega\rangle\left(\frac{-U_2}{U_1}\right)^{i\beta\omega/2\pi} + \langle a^\dagger_\omega a^\dagger_\omega\rangle\left(\frac{U_1}{-U_2}\right)^{i\beta\omega/2\pi}\right]$$

ω^2 from derivative cancels $\frac{1}{\omega}$ from fields

$$\delta^2(\delta\Omega)$$

comes from angular integrals

If we now set

$$\langle a_\omega a_\omega \rangle = \frac{e^{-\beta\omega/2}}{1-e^{-\beta\omega}}$$

$$\langle a_\omega^+ a_\omega^+ \rangle = \frac{e^{-\beta\omega/2}}{1-e^{-\beta\omega}}$$

the integral becomes

$$\langle \partial_{U_1}\phi_1 \; \partial_{U_2}\phi_2 \rangle = N \frac{\delta^2(\delta\Omega)}{U_1 U_2} \int d\omega \; \omega \left(\frac{U_1}{-U_2}\right)^{i\beta\omega/2\pi} \frac{e^{-\beta\omega/2}}{1-e^{-\beta\omega}} \, d\omega$$

For $|u_1| > |u_2|$ we can complete the contour in the upper-half plane. We now get a minus sign with each pole that cancels the minus sign in the u-term

$$\langle \partial_{u_1} \phi_1 \, \partial_{u_2} \phi_2 \rangle \propto \frac{\delta^2(\delta\Omega)}{u_1 u_2} \frac{u_2/u_1}{\left(1 - \frac{u_2}{u_1}\right)^2}$$

$$= \frac{\delta^2(\delta\Omega)}{(u_1 - u_2)^2}$$

exactly as we needed!

This leads to a heuristic understanding of Hawking radiation.
At the horizon, we pair create particles with +ve and −ve energy. The +ve energy particle escapes to infinity while the −ve energy particle falls into the B.H. reducing its mass.

The correct mathematical statement is:
In the Unruh vacuum we have a positive outward flux of energy at ∞ and a negative flux into the horizon. The modes outside and inside are entangled.

So far we have analyzed the two point function of derivatives of the scalar field.

In fact, the stress-tensor is closely related.

Candelas has computed the stress-tensor at the horizon and at ∞. The relevant terms are

$$T_{\mu\nu} \, l^{\mu} l^{\nu} = \frac{-L}{4\pi r^2} \quad , \; L = \frac{1}{2\pi} \int d\omega \; \omega \; \frac{\sum_{l} (2l+1) \, |A_{l,\omega}|^2}{e^{\beta\omega} - 1}$$

where $l = \frac{\partial}{\partial V}$ at the horizon is the future directed normal

at $\mathscr{I}^+$ $\quad T_{\mu\nu} \, l_{\infty}^{\mu} l_{\infty}^{\nu} = \frac{L}{4\pi r^2}$, where $l_{\infty} = -\frac{\partial}{\partial u}$ is the future directed normal at $\mathscr{I}^+$.

Now we note something important. The flux through the horizon measured along a future-directed null vector is negative.

So the null energy condition is violated.

This is because of quantum effects. Due to quantum-mechanical fluctuations, the classical energy conditions need not be satisfied.

This negative flux corresponds to a steady decrease in area and mass.

The area theorem does not hold because the null-energy condition is violated.

Second, note that the black hole emits in all modes → photons, gravitons and other particles as well.
The dominant emission is in massless particles and particles below $\frac{1}{\beta}$.

In particular this means the B.H. loses mass at

$$\frac{dM}{dt} = -\sigma T^4 A = -\frac{\sigma}{M^2}$$

in a time $t_{life} \propto M^3$, the b.h. evaporates completely.

This brings us to the OLD INFORMATION PARADOX.

Consider a black hole formed from collapse.
If this b.h. is isolated from the rest of the Universe, we can think of it as a pure state $|\Psi\rangle$.

Then if we trust the Hawking radiation calculation then in the end we have a thermal collection of uncorrelated particles. In fact the final state on $\mathscr{I}^+$ looks like a density matrix.
This suggests that we have

pure state $\longrightarrow$ mixed state??

To berate an obvious point, the Schrödinger equation is

$$i \frac{d |\psi\rangle}{dt} = H |\psi\rangle.$$

which is reversible.

Hawking radn seems to be irreversible.

Most naively, we might try to correct the Hawking rad formula to preserve information. However within perturbation theory this seems problematic.

This is because information does not stay at the horizon → black holes have no hair.

Regardless of the details of the initial collapse, the final geometry approaches a universal geometry.

In fact any structure we put on the horizon tends to decay exponentially.

One possibility is that the information doesn't come out at all. Of course, we don't know what happens once we reach a Planck-mass sized black hole.

But if we trust the Hawking computation down to this level then all the information must be stored in this remnant.

[Explain problems with density of states.]

Remnants also create a problem with naturalness. Recall that a massive particle of mass M propagating over a distance r leads to a force

$$F_{\text{Yukawa}} = \frac{e^{-Mr}}{r}$$

The force between two particles exchanging virtual remnants is small, but becomes huge due to the number of these.