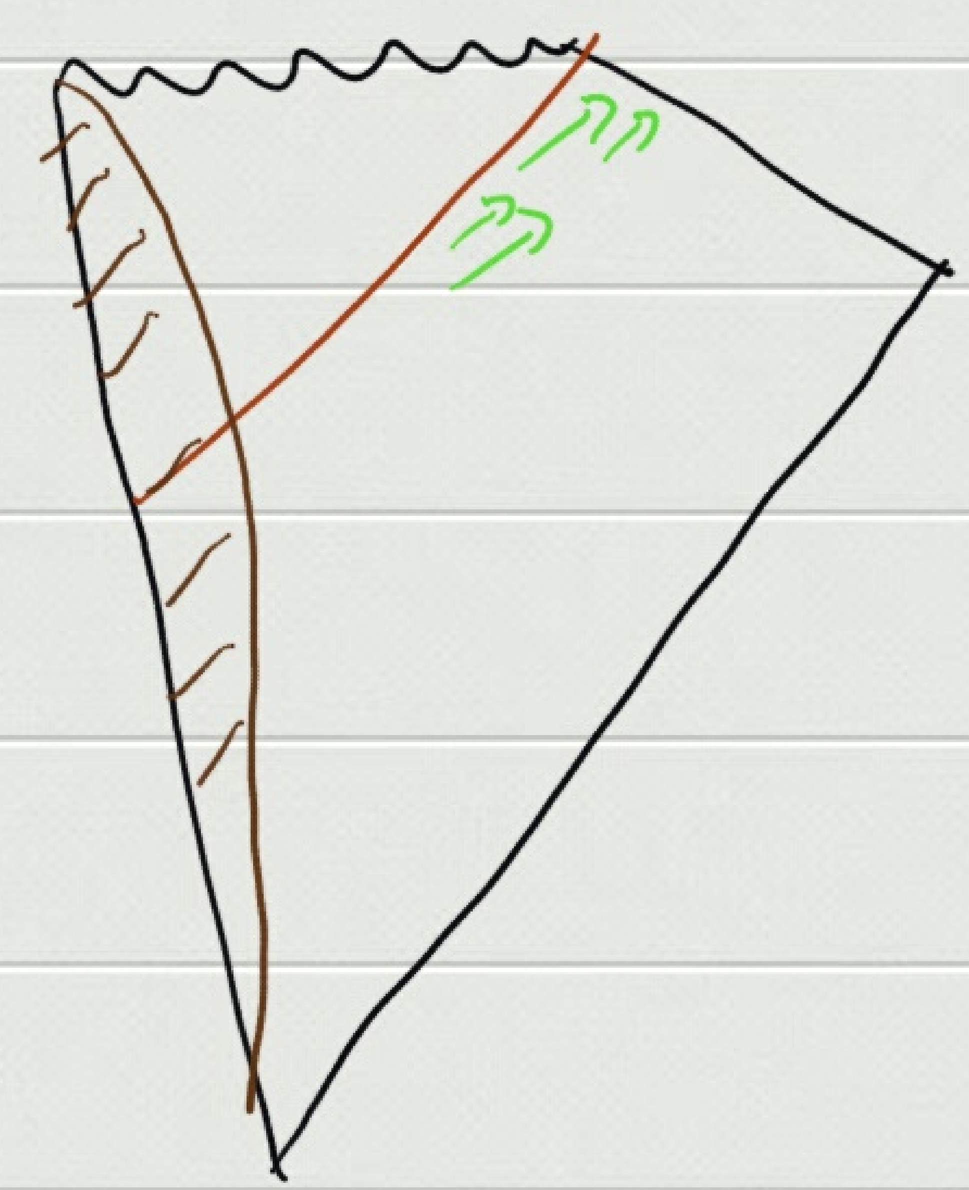
# Quantum Aspects of Black Holes - Lecture 18 - 25 October 2016

Last time we discussed the naive information paradox.

The paradox is as follows.

Consider a black-hole formed from collapse



The Hawking radiation seems to emerge from the horizon which seems to have no information

about the infalling matter.
eg. we derived

$$\langle a_\omega^{+} a_{\omega'} \rangle = \frac{e^{-\beta\omega}}{1-e^{-\beta\omega}} \delta(\omega-\omega')$$

For the modes outside independent of the details of the collapse.

This suggests the black hole evaporates in a time $t_{bh} \propto M^3$.

If the final radiation is thermal, then we have a contradiction. The Schrodinger equation

$$\frac{i\partial|\psi\rangle}{\partial t} = H|\psi\rangle$$

does not allow a pure state to turn into a thermal state.

But this is not a paradox because

1) Pure states can mimic mixed states very closely.

2) Hawking's calculation is not precise enough to distinguish between a final pure state and final mixed state.

To understand claim 1, consider a closed system where the Hilbert space has dimension $e^S$.

This means that a basis for the Hilbert space is given by

$$|1\rangle \; \ldots \; |e^S\rangle .$$

But a typical state is given by

$$|\psi\rangle = \sum a_i |i\rangle \quad \text{with} \quad |a_i|^2 = 1$$

So the Hilbert space can be identified with

$$\mathbb{C}P^{e^S}$$

We now want to understand observables on typical states in this Hilbert space. This means that we pick a random pure state using a democratic probability distribution, $|\psi_{typ}\rangle$, in this Hilbert space and ask for the expectation value of an operato A

$$\langle \psi_{typ} | A | \psi_{typ} \rangle$$

This is computed by averaging $\langle\psi|A|\psi\rangle$ as $|\psi\rangle$ runs over the whole Hilbert space

$$\langle\langle A\rangle\rangle = \int_{CP^{s}} \langle\psi|A|\psi\rangle \, d\mu_{|\psi\rangle}.$$

where $d\mu_{|\psi\rangle}$ is a measure factor that assigns equal weights to equal volumes on $CP^{s}$.

Such a measure is easy to find. We simply consider

$$\frac{1}{V} da_1 \ldots da_{e^s} \, \delta(|a_1|^2 + \ldots |a_{e^s}|^2 - 1) := d\mu$$

where $V$ is the normalization required to ensure

$$\int d\mu = 1$$

We can work out the normalization explicitly

$$\int \frac{e^{it(\sum_i |a_i|^2 - 1)}}{2\pi} dt \prod_i d|a_i|^2 \quad (\pi)^{e^s} e^{-\sum |a_i|^2}$$

from angular part of $a_i$

$$\int \left(\frac{-1}{it-\varepsilon}\right)^{e^s} \frac{e^{-it}}{2\pi} dt \quad (\pi)^{e^s}$$

complete $t$ contour through the lower half plane, pick up pole at $t=-i\varepsilon$ with residue

$$\frac{(\pi)^{e^s}}{\Gamma(e^s)}.$$

Now consider an operator $A$. We claim that

$$\langle\langle A \rangle\rangle = \frac{1}{e^S} \mathrm{tr}(A).$$

This means that the expectation value of $A$ in a typical state is exactly the same as in the identity density matrix $\rho = \frac{1}{e^S} I$ that assigns equal probability to all states.

This is easy to prove.

Pick a basis so $A = \sum_i A_{ii} |i\rangle\langle i|$

then

$$\langle\langle A \rangle\rangle = \int \sum_i |a_i|^2 A_{ii} \quad \frac{\delta(\sum |a_i|^2)}{V} \, d^2 a_i$$

$$= \sum_i A_{ii} \langle |a_i|^2 \rangle .$$

Since $\langle |a_i|^2 \rangle = \alpha^2$ independent of $i$ and since

$$\langle \sum |a_i|^2 \rangle = e^S \alpha^2 = 1$$

we must have

$$\alpha^2 = e^{-S}$$

$$\Rightarrow \langle\langle A \rangle\rangle = e^{-S} \sum_i A_{ii} = e^{-S} \, \mathrm{tr}(A) .$$

This is a simple, but it means that if I hand you a pure state you will get the same $\langle A \rangle$ as in the mixed state.

But maybe pure states can be distinguished by fluctuations? i.e. perhaps the average over pure states is $\frac{\mathrm{tr}(A)}{e^S}$ but individual pure states see large fluctuations from this value?

The next theorem rules this out.

<u>Claim:</u> Consider some operator $A$. Then

$$\int \left( \langle \psi | A | \psi \rangle - \frac{1}{e^S} \operatorname{tr}(A) \right)^2 d\mu.$$

$$= \frac{1}{(e^S+1)} \left[ \frac{\operatorname{tr}(A^2)}{e^S} - \frac{(\operatorname{tr}(A))^2}{e^S} \right]$$

So the average deviation is suppressed by $e^{-S/2}$!

Proof: In the same basis, we now consider

$$\int d\mu \left( \sum_i A_{ii} |a_i|^2 - \frac{1}{e^S} \sum A_{ii} \right)^2$$

$$= \sum_{i,j} \int d\mu \left( A_{ii} A_{jj} |a_i|^2 |a_j|^2 + \frac{1}{e^{2S}} \sum A_{ii} A_{jj} \right.$$

$$\left. - 2 \frac{A_{ii} |a_i|^2}{e^S} A_{jj} \right)$$

$$= \sum_{i,j} \int d\mu \; A_{ii} A_{jj} \; |a_i|^2 |a_j|^2 - \left( \frac{\sum A_{ii}}{e^S} \right)^2$$

$$\int \frac{\prod d|a_i|^2}{V} e^{-\Sigma|a_i|^2} e^{it(\Sigma|a_i|^2-1)} |a_i|^4 = \frac{2\int \frac{e^{-it}}{(it-\varepsilon)^{e^s+2}}}{\int \frac{e^{-it}}{(it-\varepsilon)^{e^s}}}$$

$$= \frac{2}{e^s(e^s+1)}$$

and $\int \frac{\prod d|a_i|^2}{V} e^{-\Sigma|a_i|^2} e^{it(\Sigma|a_i|^2-1)} |a_i|^2 |a_j|^2 = \frac{1}{e^s(e^s+1)}$

for $i \neq j$

So the original integral becomes

$$\sum_i A_{ii} A_{jj} \left( \frac{2\,\delta_{ij}}{e^s(e^s+1)} + \frac{(1-\delta_{ij})}{e^s(e^s+1)} \right) - \left( \frac{\sum A_{ii}}{e^s} \right)^2$$

$$= \sum_i \frac{A_{ii}^2}{e^s(e^s+1)} - \left(\sum A_{ii}\right)^2 \left( \frac{1}{e^s(e^s+1)} - \frac{1}{e^{2s}} \right)$$

$$= \frac{\mathrm{tr}(A^2)}{e^s(e^s+1)} - \frac{\mathrm{tr}(A)^2}{e^{2s}(e^s+1)} \quad \text{as claimed!}$$

This is a general result valid for a statistical system.
If the system is in a pure state with energy

$$E \in [E_0 - \Delta E,\ E_0 + \Delta E]$$

then there are $e^S$ states in this interval.

A typical linear combination of energy eigenstates in this interval is indistinguishable from the microcanonical mixed state.

Now we return to Hawking radiation.
We have shown that 2-pt functions have a "thermal". form. But this is misleading because pure-states look exponentially close to thermal states!

Hawking's calculation is not precise enough to lead to a paradox.

Can we improve Hawking's computation to obtain a paradox. To correct the computation we need to take the back-reaction of the radiation on the geometry into account.

the energy of typical Hawking quanta is

$$E = \frac{M_{Pl}^2}{M}$$

Recall the gravitational coupling constant is

$$\frac{E^2}{M_{Pl}^2} = \frac{M_{Pl}^2}{M^2} \text{ in this case.}$$

But $\frac{M_{Pl}^2}{M^2} = \frac{1}{S}$ !

So by correcting the computation perturbatively we can develop a power series in $\frac{1}{S}$

But the difference between pure and mixed states is non-perturbative in this coupling.

$(e^{-S} = e^{-1/\text{coupling}})$

And Q.F.T in curved spacetime cannot deal with such effects.

To refine the information paradox, we need other arguments!