QABH - Lecture 19, 4 Nov 2016

The last time, we discussed and resolved the naive information paradox.

The point was that just computing low-point correlation functions in Hawking radiation is insufficient to declare that the final state is thermal.

To refine the information paradox, we need some results on how fast the information comes out.

How do we quantify this?

One popularly used measure is to consider the purity of the state "outside" the black hole.

Let us consider this in a simple example. Say we are given a system, and we want to determine whether it is pure.

By making many observations (on many copies of the system) we can determine its density matrix ρ

we then masure

$$S = -\mathrm{tr}(\rho \ln \rho).$$

If $S=0$, the state is pure, otherwise it is mixed.

This is easy to see. A pure state is of the form

$$\rho = |\psi\rangle\langle\psi|$$

So in some basis, it looks like

$$\rho = \begin{pmatrix} 1 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}$$

Clearly $S = 0$ for such a state.

The maximum value that S can take is
$\log(\dim(H\text{-space}))$

A simple example. Consider a two qubit system. Take

$$|\psi\rangle = \left(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle \right) \frac{1}{\sqrt{2}}$$

then the density matrix for qubit-1 is

$$tr_2 \; |\psi\rangle\langle\psi| = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

so $-\mathrm{tr}(\rho \ln \rho) = \ln 2$.

In the state

$$|\psi\rangle = |\uparrow\uparrow\rangle,$$

the density matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and so, clearly $S = 0$.

Now we would like to apply this diagnostic to the black hole. The first step to doing this is to generalize this diagnostic to quantum field theory.

The first complication we encounter is that the vacuum state itself has infinite entanglement.

This is easiest to see by writing the vacuum as a wave-function on fields on a time slice.

Recall that we have

$$\Phi(0,x) = \int \left(a_k e^{ik\cdot x} + a_k^{\dagger} e^{-ik\cdot x}\right) \frac{d^3 k}{\sqrt{2|k|}\,(2\pi)^{3/2}}$$

$$\Pi(0,x) = -i\int \left(a_k e^{ik\cdot x} - a_k^{\dagger} e^{-ik\cdot x}\right) \frac{d^3 k}{(2\pi)^{3/2}}\sqrt{\frac{|k|}{2}}$$

In particular

$$\sqrt{\frac{|k|}{2}}\int \left(\phi(0,x) + \frac{i}{|k|}\Pi(0,x)\right) e^{-ik\cdot x} \frac{d^3 x}{(2\pi)^{3/2}} = a_k$$

If we define

$$\phi_k = \int \phi(0,x)\, e^{ik\cdot x} \frac{d^3x}{(2\pi)^{3/2}}$$

and $$\pi_k = \int \pi(0,x)\, e^{ik\cdot x} \frac{d^3x}{(2\pi)^{3/2}}$$

then

$$[\phi_k, \pi_{k'}] = i \int e^{ik\cdot x + k'x} \frac{d^3x}{(2\pi)^3}$$

$$= i \quad \delta(k+k').$$

so we can realize $\pi_{k'} = -i \dfrac{\partial}{\partial \phi_{-k'}}$

The equation

$$a_k |\Omega\rangle = 0$$

can then be written as

$$\left(\phi_k + \frac{1}{|k|} \frac{\partial}{\partial \phi_{-k}} \right) \psi[\phi_k] = 0$$

which is solved by

$$\psi[\phi_k] = \int e^{-\frac{1}{2} \int |k| \phi_k \phi_{-k} \, d^3k}$$

converted to position space, we can write

$$\Psi[\phi] = e^{-\int \phi(x,0)\,\phi(x',0)\,G(x,x')\,d^3x\,d^3x'}$$

where $$G(x,x') = \int |k|\, e^{ik\cdot(x-x')}\, d^3k.$$

We see from the expression above that just the ground state is highly entangled.

One possibility is just to subtract off this entanglement. So for a region R we consider

$$S(R, |\Psi\rangle) - S(R, |\Omega\rangle)$$

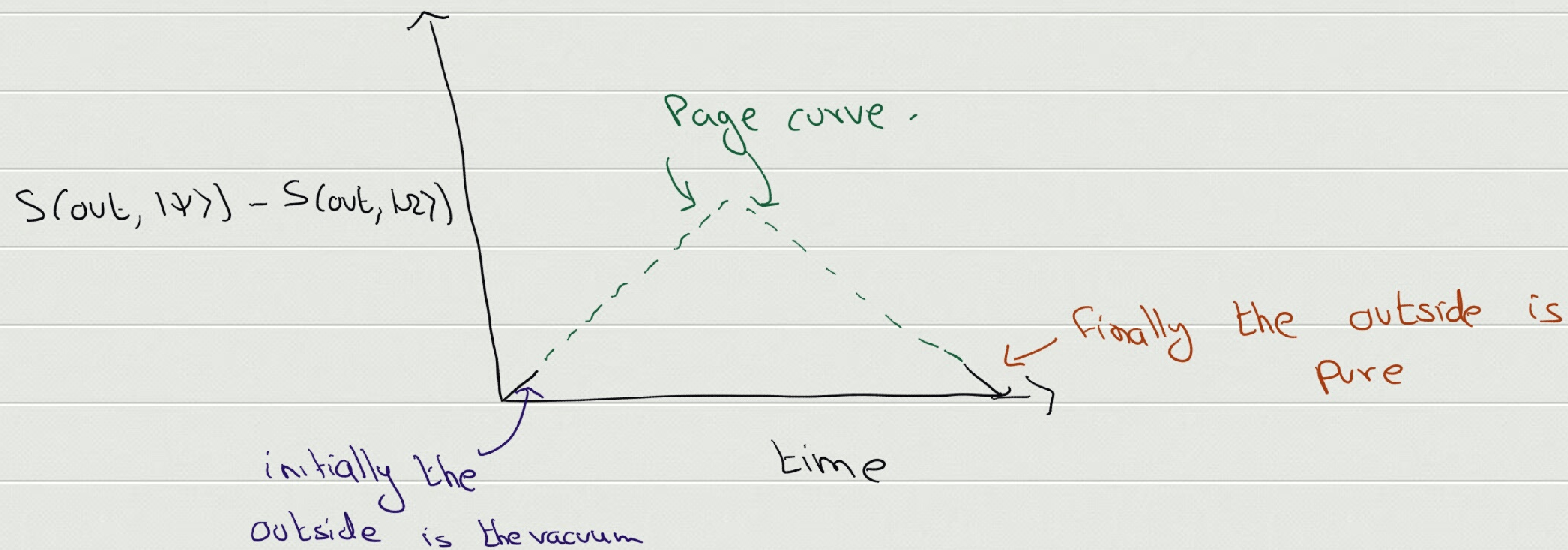
If this is 0, we say that the matter in region R is in a pure state

In theories of gravity, no one has succeeded in defining local entanglement entropy precisely.

This is because there are no gauge-invariant local operators in gravity.

Whenever we put a lump of matter in a region, its influence stretches out to infinity by the Gauss law! [explain more]

For now, we will neglect this complication. Then consider the entanglement entropy of the Hawking radiation with the black hole.



The Page curve is derived by considering the process of breaking up a large system into two subsystems. ← [Explanation in class!]

Lets get some intuition for this. Say we have a big system whose Hilbert space has dimension

$$n\,m.$$

We divide it into a space with dimension m and another with dimension n.

Now we consider a typical state in the big space. What is the entanglement entropy of the m-subsystem?

Let $|M_i\rangle$, $i = 1 \ldots m$ be a basis for one subsystem

and $|N_j\rangle$, $j = 1 \ldots n$ be the basis for the other

Assume without loss of generality that $m \leq n$

then a general state has the form

$$|\psi\rangle = \sum_{i,j} A_{ij} \, |M_i\rangle \, |N_j\rangle .$$

Here A is a rectangular matrix with m-rows and n-columns.

By a suitable change of basis in both subsystems we can diagonalize A. So that

$$A_{ij} = \sum_{m} U_{im} D_{mm} V_{mj}$$

where U is a $m \times m$ unitary and V is a $n \times n$ unitary. and D has the form

$$m\left\{\begin{pmatrix}\underbrace{\begin{matrix} d_{11} & & & & \\ & d_{22} & & & \\ \cdots & & d_{33} & & \\ \cdots & & & \ddots & \\ \cdots & & & & d_{mm}\end{matrix}}_{m} & \underbrace{\begin{matrix}\cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots\end{matrix}}_{n-m}\end{pmatrix}\right. \qquad \cdot \equiv 0$$

The density matrix of the m-subsystem is then

$$\rho = \sum |d_{ii}|^2 \, |\tilde{M}_i\rangle\langle\tilde{M}_i|$$

with entropy

$$S = \sum |d_{ii}|^2 \ln |d_{ii}|^2$$

Since we do not expect any particular direction in the Hilbert space to be favoured

$$\langle |d_{ii}|^2 \rangle = \frac{1}{m} \Rightarrow \langle S \rangle = \ln m$$

where the average is with respect to all possible pure states in the full system.

We are also interested in showing that deviations from this value are small. To do this we write

$$\rho = \frac{1}{m} I + \delta$$

$$\ln \rho = \ln (I + m\delta) - \ln m = m\delta - \ln m - \frac{m^2 \delta^2}{2}$$

$$-\mathrm{tr}\,(\rho \ln \rho) = \ln m - \frac{m}{2} \mathrm{tr}\, \delta^2$$

[Linear terms in δ vanish because $\mathrm{tr}\, \delta = 0$

Factor of $\frac{1}{2}$ due to partial cancellation of quadratic terms.]

We now show that this deviation is suppressed by the dimension of the Hilbert space

Note that if p_i are the eigenvalues of ρ then

$$\mathrm{tr}(\delta^2) = \sum_i \left(p_i - \frac{1}{m}\right)^2 = \sum p_i^2 - \frac{1}{m} \quad \leftarrow = \sum \frac{1}{m^2}$$

So we need to estimate.

$$\sum p_i^2 = \mathrm{tr}(\rho^2).$$

But

$$\rho_{i\tilde{\imath}} = \sum_j A_{ij} A^*_{\tilde{\imath}j}$$

$$\mathrm{tr}(\rho^2) = \sum_{i,\tilde{\imath},j,k} A_{ij} A^*_{\tilde{\imath}j} A^*_{ik} A_{\tilde{\imath}k}$$

The result is $\sum p_i^2 = \dfrac{m+n}{mn+1}$

So $\langle \sum p_i^2 \rangle - \dfrac{1}{m}$ $\left(= \sum \dfrac{1}{m^2}\right)$ $= \dfrac{(m+n)\,m - (mn+1)}{(mn+1)\,m}$

$= \dfrac{m^2-1}{m(mn+1)}$

Therefore we see that

$$S \approx \ln m - \frac{(m^2-1)}{2m(mn+1)} \approx \ln m - \frac{m}{2n}$$

Now note that if we have a very large Hilbert space, then the dimension and entropy are related through

$$m = e^{S_1} \; ; \quad n = e^{S_2} \; ; \quad mn = e^{S_1+S_2}$$

and $\quad S_2 > S_1$

so

$$S_{ent} \approx S_1 - \frac{1}{2} e^{(S_1 - S_2)}$$

The second term is always smaller than 1 and in the thermodynamic limit, the first term dominates.