

Quantum Aspects of Black Holes - Lec 21

Last time, we discussed the "cloning paradox". Before we proceed, it is useful to consider the issue of the squeezing of the nice slices.

Recall the following picture for the nice slices



Now consider an infalling observer. The metric inside is also

$$ds^2 = \left(\frac{2m}{r} - 1\right) dr^2 - \left(\frac{2m}{r} - 1\right) dt^2 - r^2 d\Omega^2$$

An observer who desires the maximum possible proper time before hitting the singularity moves only on r [not t or Ω], so has

$$d\tau = -\sqrt{\frac{2m}{r} - 1} dr$$

$$\text{or } \tau = \int_0^{2m} \sqrt{\frac{2m}{r} - 1} dr = m\pi.$$

On the other hand if we consider the Oppenheimer-Snyder collapse then recall that the metric is

$$ds^2 = -d\tau^2 + (1 - \sqrt{6\pi P_0} \tau)^{4/3} dr^2 + r^2 (1 - \sqrt{6\pi P_0} \tau)^{4/3} d\Omega^2$$

for $r < r_s$ [see lecture 8.]

and we also have

$$m = \frac{4}{3} \pi r_s^3 P_0$$

The point of horizon crossing of the surface of the shell is

$$r_s^2 (1 - \sqrt{6\pi P_0} \tau)^{4/3} = 4m^2$$

$$\text{or} \quad 1 - \sqrt{6\pi P_0} \tau = \left(\frac{2m}{r_s} \right)^{3/2}$$

the point at which the surface of the shell hits the singularity is

$$\tau = \frac{1}{\sqrt{6\pi P_0}}$$

the difference in times is

$$\begin{aligned} \delta \tau &= \left(\frac{2m}{r_s} \right)^{3/2} \frac{1}{\sqrt{6\pi P_0}} = \sqrt{\frac{(2m)^3}{6\pi P_0 r_s^3}} \\ &= \frac{4m}{3} \end{aligned}$$

When we restore factors of M_{Pl} , we find that the ratio between the proper time of a nice slice observer inside the l.h. and one outside the l.h. is:

$$\left(\frac{M}{M_{Pl}}\right)^2$$

In particular if we consider an excitation with energy E locally on the nice slice, the energy w.r.t. the asymptotic Hamiltonian is $E \left(\frac{M_{Pl}}{M}\right)^2$.

Strong Subadditivity

First, we will review the notion of strong subadditivity. This is a very non-trivial theorem concerning von-Neumann entropies.

First some notation. Consider a Hilbert space with **three** factors.

$$H = H_A \otimes H_B \otimes H_C$$

Then we denote the state of the full system as

$$\rho_{ABC}$$

may or may not be pure

We also denote

$$\rho_{AB} = \text{tr}_C(\rho_{ABC})$$

$$\rho_A = \text{tr}_B(\rho_{AB})$$

and similarly, we define ρ_{AC} , ρ_{BC} , ρ_B , ρ_C .

Related to these, we can define S_{ABC} , S_{AB} , S_{AC} , S_{BC} etc.

The strong subadditivity theorem states

$$S_{ABC} + S_B \leq S_{AB} + S_{BC}.$$

This is quite an important result, and also somewhat difficult to prove.

The paper [quant-ph/0408130](#) gives some intuition and a relatively simple proof.

To gain some intuition, we can consider the conditional entropy

$$S(A|B) = S_{AB} - S_B$$

Classically, the conditional entropy may be interpreted as our uncertainty about A , given B .

$$S_{cl}(A|B) = \sum_v P(v) S(A|B=v)$$

$$\text{But } S_{cl}(A|B=v) = \sum_a p(a|v) \ln(p(a|v))$$

$\uparrow$
 $p(A=a|B=v)$

$$= \sum_a \frac{p(a,v)}{p(v)} \ln \frac{p(a,v)}{p(v)} \leftarrow \text{Bayes thm: } p(a|v) = \frac{p(a,v)}{p(v)}$$

$$= \frac{1}{p(v)} \left(\sum_a p(a,v) \ln(p(a,v)) - p(v) \ln p(v) \right).$$

so

$$S_{cl}(A|B) = \sum_{a,v} p(a,v) \ln(p(a,v)) - \sum_v p(v) \ln(p(v))$$

Then strong subadditivity can be written as

$$S(A|BC) \leq S(A|B)$$

"uncertainty about A given B & C is less than the uncertainty given only B"

Omit in class

The second notion we need to introduce is the **relative entropy**. This is defined as

$$S(p \parallel \sigma) = \sum (p \ln p - p \ln \sigma)$$

No minus sign!

$$S(p \parallel \sigma) \geq 0 \quad [= 0 \text{ if } p = \sigma]$$

and it measures the "distance between p and σ "

The conditional and relative entropies are related through

Omit in class

$$S(B|A) = \ln d_B - S(P_{AB} \| P_A \otimes \frac{\underline{I}_B}{d_B}).$$

$$= \ln d_B + S_{AB} + \text{Tr}(P_{AB} \ln(P_A \otimes \frac{\underline{I}_B}{d_B}))$$

choose a basis so

$$P_{AB} = P_{a_i b_j} a_j b_j \text{ is diagonal}$$

$$= P_{ij} \delta_{ij}$$

Also

$$\left(P_A \otimes \frac{\underline{I}_B}{d_B}\right)_{ij} = q_i \delta_{ij} ; q_i = \sum_R P_{iR} \frac{1}{d_B}$$

and is independent of j .

Omit in class

So the second term yields
 $-S_B + \ln d_B$

confirming that

$$S(B|A) = \ln d_B - S(P_{AB} \parallel P_A \otimes \frac{\underline{I}_B}{d_B}). \quad (I)$$

Then SSA can be written as

$$S(P_{AB} \parallel \frac{\underline{I}_A}{d_A} \otimes P_B) \leq S(P_{ABc} \parallel \frac{\underline{I}_A}{d_A} \otimes P_{Bc})$$

Note flip in sign due to -sign in (I)

Omit in class

Strong subadditivity can then be proved using the monotonicity of relative entropy

$$S(P_A \parallel \sigma_A) \leq S(P_{AB} \parallel \sigma_{AB})$$

We will only prove the classical version of this result. Classically

$$S(P_A \parallel \sigma_A) = \sum (r_j \ln r_j - r_j \ln s_j)$$

$$S(P_{AB} \parallel \sigma_{AB}) = \sum_{j,R} r_{jR} (\ln r_{jR} - \ln s_{jR})$$

with

$$\sum_R r_{jR} = r_j$$

Omit in class

The difference is just

$$\sum_{j,k} r_{jk} \ln \left(\frac{r_j s_{jk}}{r_{jk} s_j} \right) \leq \sum_{j,k} r_{jk} \left(\frac{r_j s_{jk}}{r_{jk} s_j} - 1 \right)$$

$$= \sum_j (r_j - r_j) = 0.$$

quantum mechanically the main problem is to
bring everything into one logarithm which is
hard if p, σ do not commute.

So finally we have the result

$$S_{ABC} + S_B \leq S_{AB} + S_{BC}.$$

We can use this to prove a more useful related result. Let us add a L^{th} system D so that ρ_{ABCD} is **pure**. This is always possible. The density matrix

$$\rho = \sum_i |a_i|^2 |i\rangle\langle i|$$

can be obtained from

$$|\psi\rangle = \sum_i a_i |i\rangle |\tilde{i}\rangle$$

where $|\tilde{i}\rangle$ is an auxiliary system.

Also note that is in this basis

$$P = \tilde{P}$$

and as a basis independent statement

$$S = \tilde{S}$$

From SSA we find

$$S_D + S_B \leq S_{CD} + S_{BC}.$$

We can now forget about A and relabel

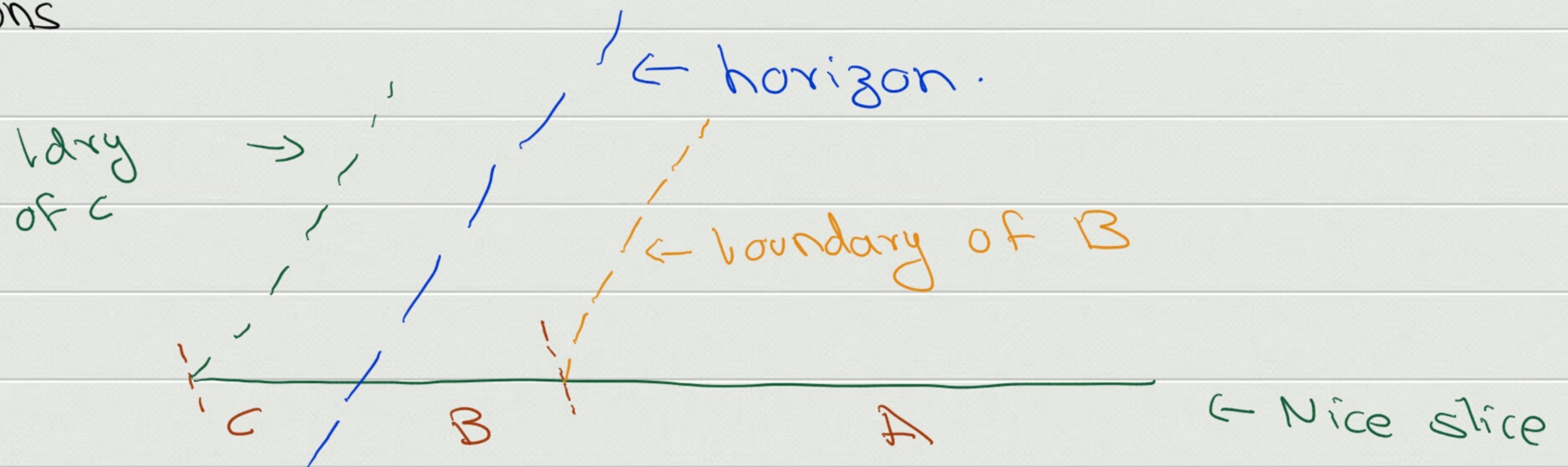
$$B \rightarrow A, C \rightarrow B, D \rightarrow C$$

$$S_{AB} + S_{BC} \geq S_A + S_C$$

The Strong Subadditivity Paradox

We now turn to the strong subadditivity paradox first formulated by Mathur and then popularized by Almheiri et. al.

Given a black hole, we divide it into 3-regions



C is the region "just behind" the horizon.

B is the region "just outside" the horizon

A is the rest of the slice.

In what follows, we neglect the entropy of the vacuum, although if the boundaries are at spheres of constant r [as shown], we can imagine that this is a constant that does not change in the dynamic analysis we undertake below.

Now let us consider S_A as a function of time.



Now all 3-regions A, B, C are populated with Hawking radiation.

But the Hawking radiation moves across boundaries.

What is in B entirely crosses over to A after a while.

So we can write $S_A(t + \delta t) = S_{AB}$

where δt is the "light-crossing" time for B.

Now if we are **after** the Page time, this means

$$S_{AB} < S_A.$$

Second notice that all times the Hawking radiation in B and C is entangled.

We expect that the near-horizon state is well-approximated by

$$e^{e^{-\beta\omega/2} a_{\omega}^{\dagger} \tilde{a}_{\omega}^{\dagger}} | \Omega \rangle$$

where $a_{\omega}^{\dagger}$, $\tilde{a}_{\omega}^{\dagger}$ are creation operators in front of, and behind the horizon.

This means that

$$S_{Bc} < S_c \approx S_B$$

Both C and B are approximately thermal when viewed independently but almost pure when viewed together.

The previous deductions are in contradiction with SSA

$$S_{AB} + S_{BC} \geq S_A + S_C$$

But here we have [after Page time]

$$S_{AB} < S_A$$

$$S_{BC} < S_C$$

Moreover, this is an order-1 contradiction that looks difficult to resolve using small corrections

This paradox is closely related to the cloning paradox.

SSA can be understood as "monogamy of entanglement".

If B is entangled with A
it cannot be entangled with C .

Here B is entangled with A [due to unitarity] and with C [due to smoothness of the horizon.]

The fact that B is entangled both with A & C is similar to the paradox that quantum information is cloned on A & C.

Some caveats

The SSA-paradox looks precise but we have been sloppy in several places.

a) we did not take the vacuum contribution into account

1) how do we fix the boundaries of A, B, C when the geometry is evolving due to back-reaction due to Hawking radiation.

How do we know that effects due to the changing boundaries do not resolve the contradiction.

↑
[Explain more in class]

c) the main caveat we will focus on is that, in gravity,

A, B, C are
not independent

The Hilbert space does not factorize
and this resolves the SSA-paradox.

d) For completeness, we also explain the Firewall / Fuzzball resolution to SSA.

This is the idea that $S_{BC} > S_C$.

But if so then regions B, C are not entangled. Then as we consider the 2-pt correlator of fields

$$\langle \phi(x_{in}) \phi(x_{out}) \rangle \neq \frac{1}{|x_{in} - x_{out}|^2}$$

If the short-distance behaviour of correlators is changed, then the horizon must have a singular geometry
 $\Rightarrow$ FIREWALL!

