QABH - Lecture 8

14 Sept 2016

Last time, we discussed the Schwarzschild black hole

$$ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \frac{dr^2}{\left(1-\frac{2m}{r}\right)} + r^2 d\Omega^2$$

Things to remember

a) red-shift exists in this geometry, and becomes ∞ between an observer at the horizon and at asymptotic infinity

b) the horizon is not a locally distinguished region. For a big b.h., curvatures are arbitrarily weak.

3) the geometry can be extended to the entire Kruskal region.

The last topic to cover is the Penrose diagram. This helps visualize the causal structure by compactifying the spacetime. define

$$x = \tan^{-1} U; \qquad \xi = \tan^{-1} V$$

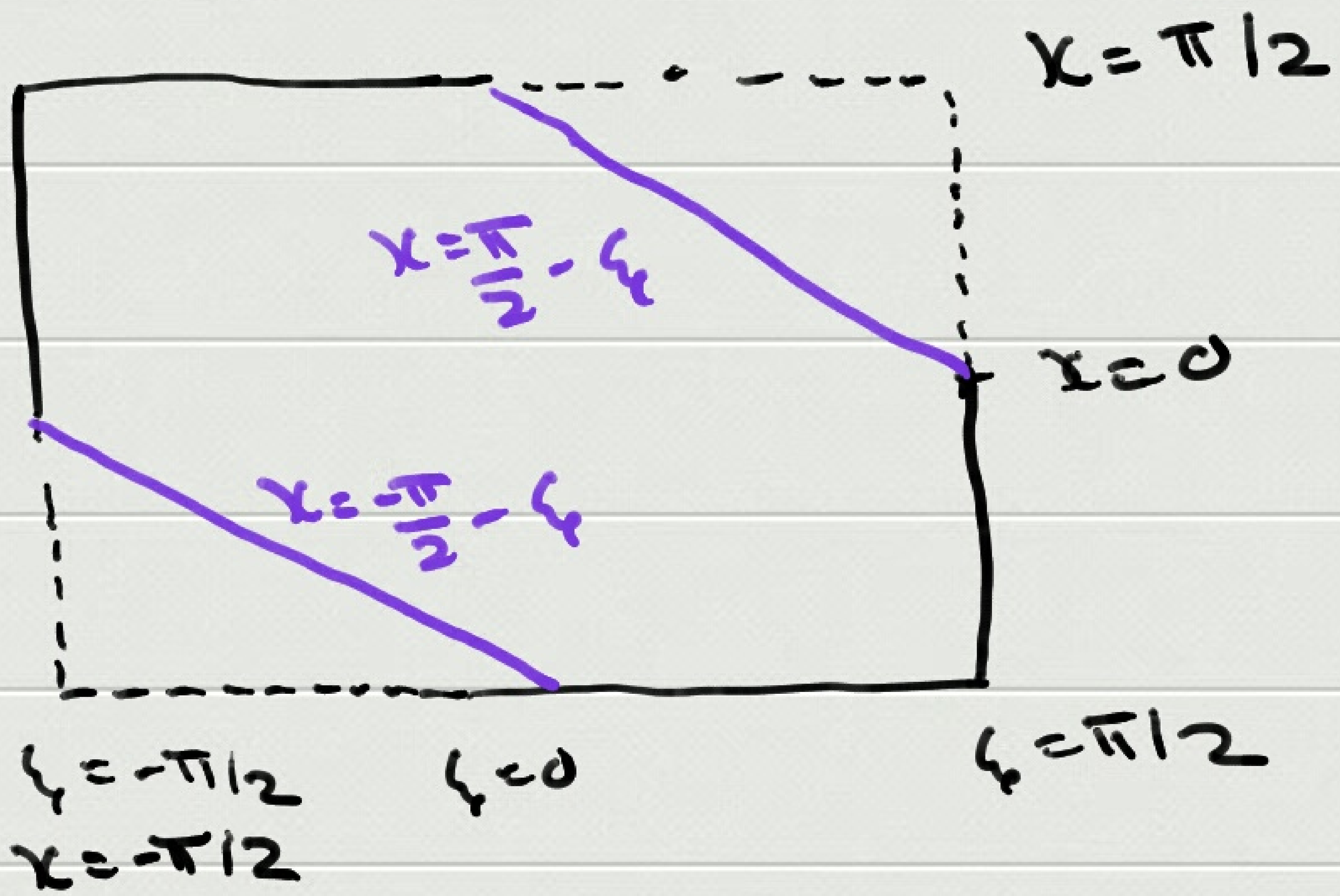
then $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$; $\xi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

except for the cutoff at $UV = \underline{1}$ or

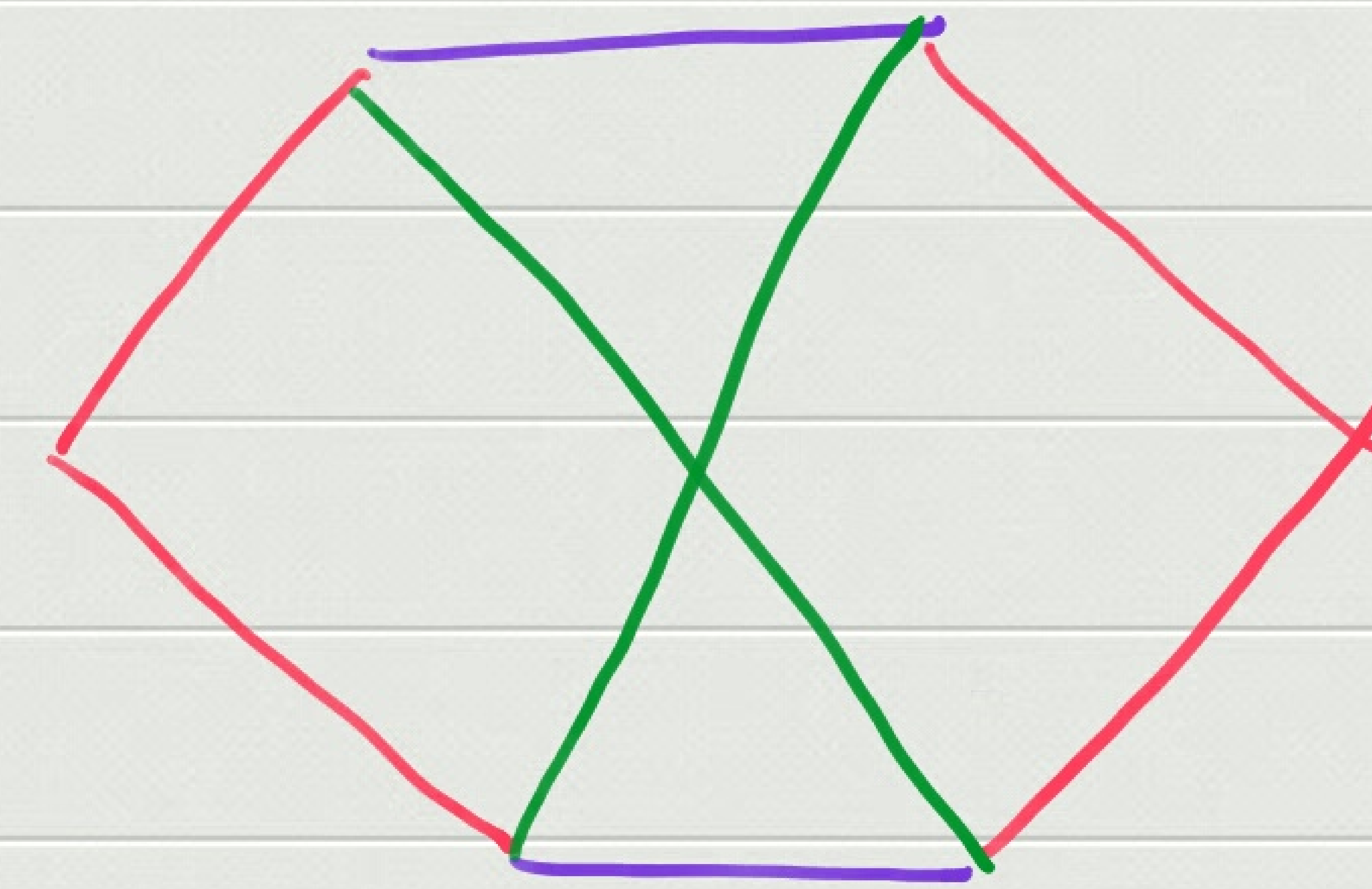
$\tan x \tan \xi = 1$ or $\xi = \frac{\pi}{2} - x$

or $\xi = -(\frac{\pi}{2} + x)$.

leads to

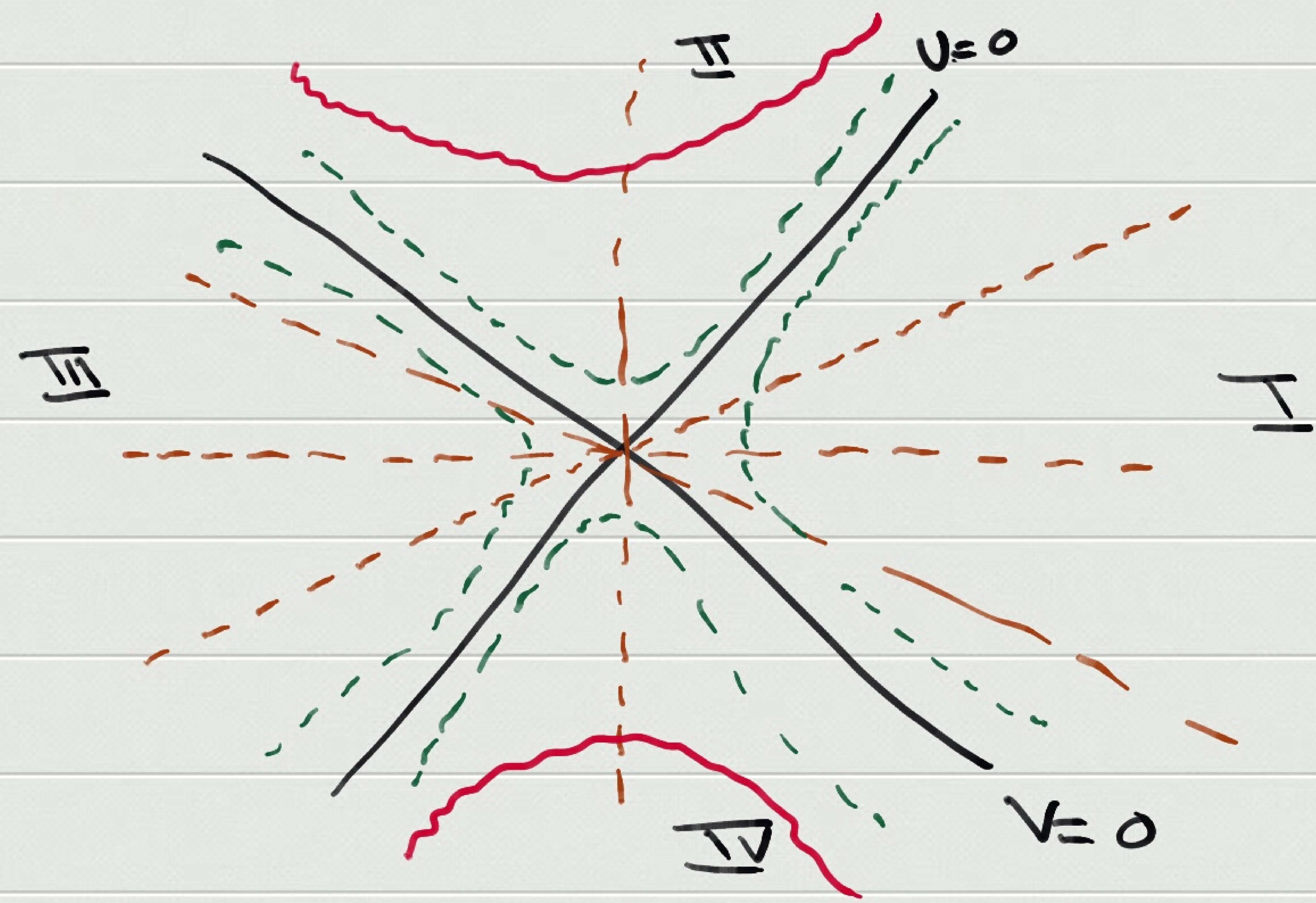


Rotate 45° $\longrightarrow$
to make nullrays at 45°

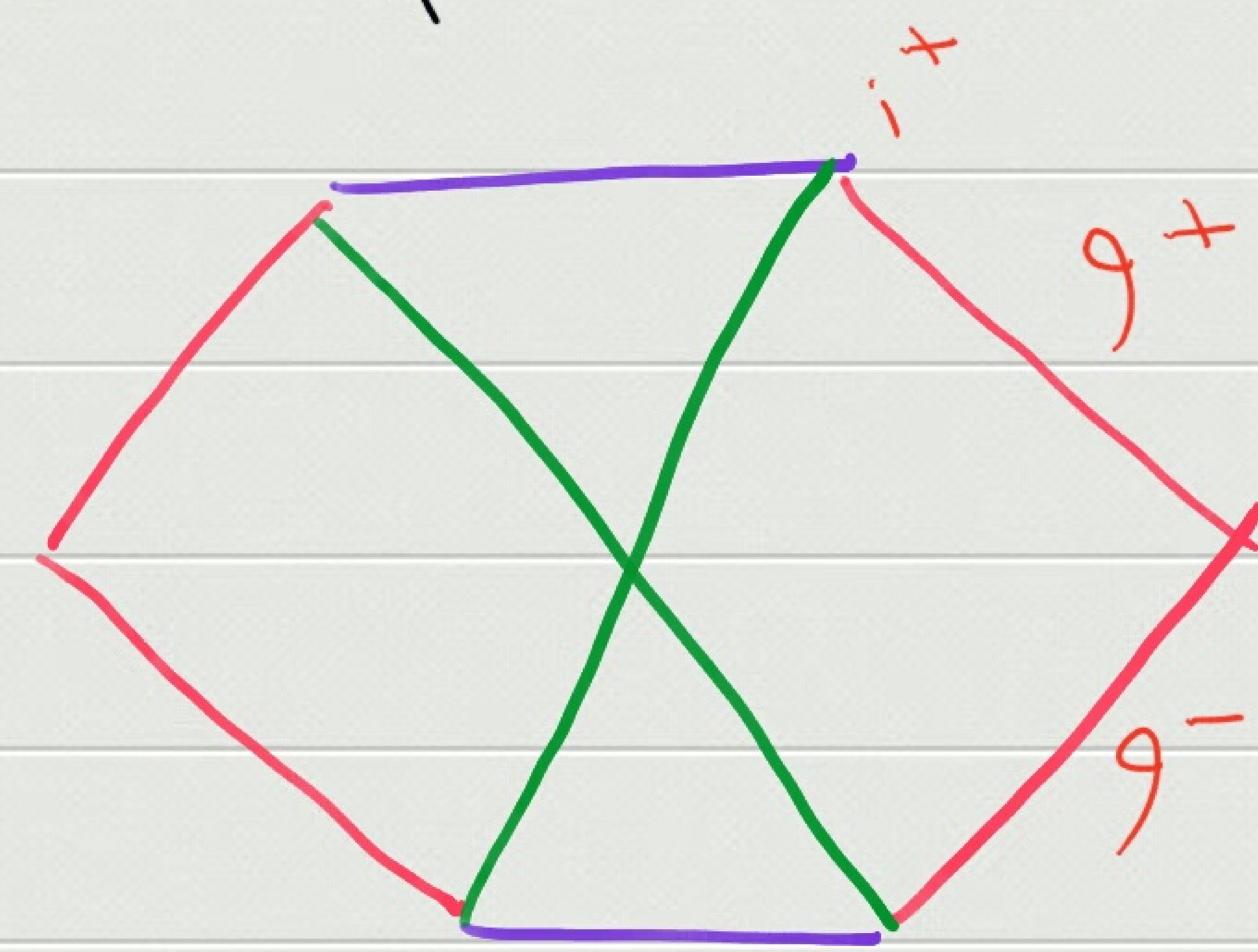


The green lines are <u>horizons</u>. They divide the spacetime into two disconnected parts

The Penrose diagram is just a trick to help us visualize the spacetime. We get much of the same information from the Kruskal diagram.



The Penrose diagram is especially useful in understanding the boundaries of the spacetime. If we think of a light ray, recall that it travels along $u =$ const or $V =$ const. So all light rays that don't fall into the b.h. end up at future null infinity, $\mathcal{I}^+$



Now think of a timelike geodesic that lives outside the B.H

For this geodesic, very far from the B.H

$$\delta r_* = \delta r < \delta t$$

so

$$(r_* - t)$$

keeps becoming more and more negative and so

$$U \to 0 \;,\; V \to \infty \quad \text{or} \quad \chi = 0, \; \xi = \frac{\pi}{2}$$

This point is future null infinity, i^+

Uniqueness of the Schwarzschild solution – Birkhoff's theorem

The Schwarzschild solution is important because it is unique as a vacuum solution with spherical symmetry. This is called Birkhoff's theorem and easy to derive.

Spherical symmetry means the metric is

$$ds^2 = -A_2(r,t)\,dt^2 + C_2(r,t)\,dr\,dt + B_2(r,t)\,dr^2 + D_2(r,t)\,\underbrace{d\Omega^2}_{d\theta^2+\sin^2\theta\, d\phi^2}$$

sign of spherical symmetry

we now just define a new "r" coordinate,

$$r_{new} = D_2(r,t)$$

to avoid cluttering the notation we use r for the new coordinate as well. Metric now becomes

$$ds^2 = A_1\, dt^2 + C_1\, dt\, dr + B_1\, dr^2 + r^2 d\Omega^2$$

by redefining t, we take

$$ds^2 = A\, dt^2 + B\, dr^2 + r^2 d\Omega^2$$

So far we have used only Kinematics. Now we input dynamics we compute the Ricci tensor. We find

$$R_{tt} = R_{rr}\frac{A}{B} - \frac{B\frac{\partial A}{\partial r} + A\frac{\partial B}{\partial r}}{rB^2}$$

$$R_{tr} = \frac{\frac{\partial B}{\partial t}}{rB} \; ; \quad R_{\theta\theta} = 1 - \frac{1}{B} - \frac{r\frac{\partial A}{\partial r}}{2AB} + \frac{r\frac{\partial B}{\partial r}}{2B^2}$$

$$R_{\phi\phi} = \sin^2\theta \, R_{\theta\theta}$$

Note immediately that $R_{tt} = R_{rr} = 0$

$$\Rightarrow \quad B \frac{\partial A}{\partial r} + A \frac{\partial B}{\partial r} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial r}(AB) = 0$$

Also $R_{tr} = 0 \quad \Rightarrow \quad \frac{\partial B}{\partial t} = 0$

So the equation remaining to solve is $R_{\theta\theta} = 0$

This can be solved through

$$A = -\left(1 - \frac{2m}{r}\right) ; \quad B = \frac{1}{1 - \frac{2m}{r}}$$

Footnote!

We could take $AB = f(t)$, rather than $AB = -1$ but f can be eliminated through a suitable redef. of t.

Higher dimensional analogues.

We could imagine that we are living in higher dimensions. In d spacetime dimensions, there is an analogue of the Schwarzschild solution

$$ds^2 = -\left(1 - \frac{\mu}{r^{d-3}}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{\mu}{r^{d-3}}\right)} + r^2 d\Omega_{d-2}^2$$

where $\mu = \dfrac{16\pi M}{(d-2)\,\Omega_{d-2}}$

Gravitational Collapse

So far, we have considered eternal black holes. The geometry is exactly static in the exterior. But we believe that real black holes form from collapse.

We will now consider one collapsing solution.
The techniques to analyze such solutions were analyzed
B. Datt, in Z. fur. Physik in October 1937 (thanks, Supurna for this info!)
In 1939, Oppenheimer-Snyder recognized that these solutions could be used to analyze continued gravitational collapse

We will not go through the entire derivation, but just explain some key physical points.

The simplest matter that could collapse is dust. The peculiarity of dust is that its rest energy $\propto mc^2$ is much larger than its pressure $\propto m v_{rms}^2$. So in the comoving frame of dust, the stress tensor is

$$T^{00} = \rho_0$$
$$T^{0i} = T^{ij} = 0.$$

To analyze the collapse of dust, write the metric as

$$ds^2 = -d\tau^2 + A\, dr^2 + x\, d\Omega^2$$

A solution is:

$$ds^2 = -d\tau^2 + \left(1 - \sqrt{6\pi\rho_0}\,\tau\right)^{4/3} dr^2 + r^2 \left(1 - \sqrt{6\pi\rho_0}\,\tau\right)^{4/3} d\Omega^2 \,, \quad r < r_s$$

$$T_{\tau\tau} = \frac{\rho_0}{\left(1 - \sqrt{6\pi\rho_0}\,\tau\right)^2} \,, \quad r < r_s$$

$$ds^2 = -d\tau^2 + \frac{dr^2}{h(r,\tau)^{2/3}} + r^2\, h(r,\tau)^{4/3}\, d\Omega^2 \,, \quad r > r_s$$

$$h(r,\tau) = \left(1 - \sqrt{6\pi\rho_0}\,\tau\, \frac{r_s^{3/2}}{r^{3/2}}\right)$$

$$T_{\mu\nu} = 0 \,, \quad r > r_s \,.$$

Since the soln outside is a vacuum spherically symmetric solution it must be Schwarzschild! To see this,

First set $x^2 = r^2 h^{4/3}$ or $x = r h^{2/3}$

we want "t" so that

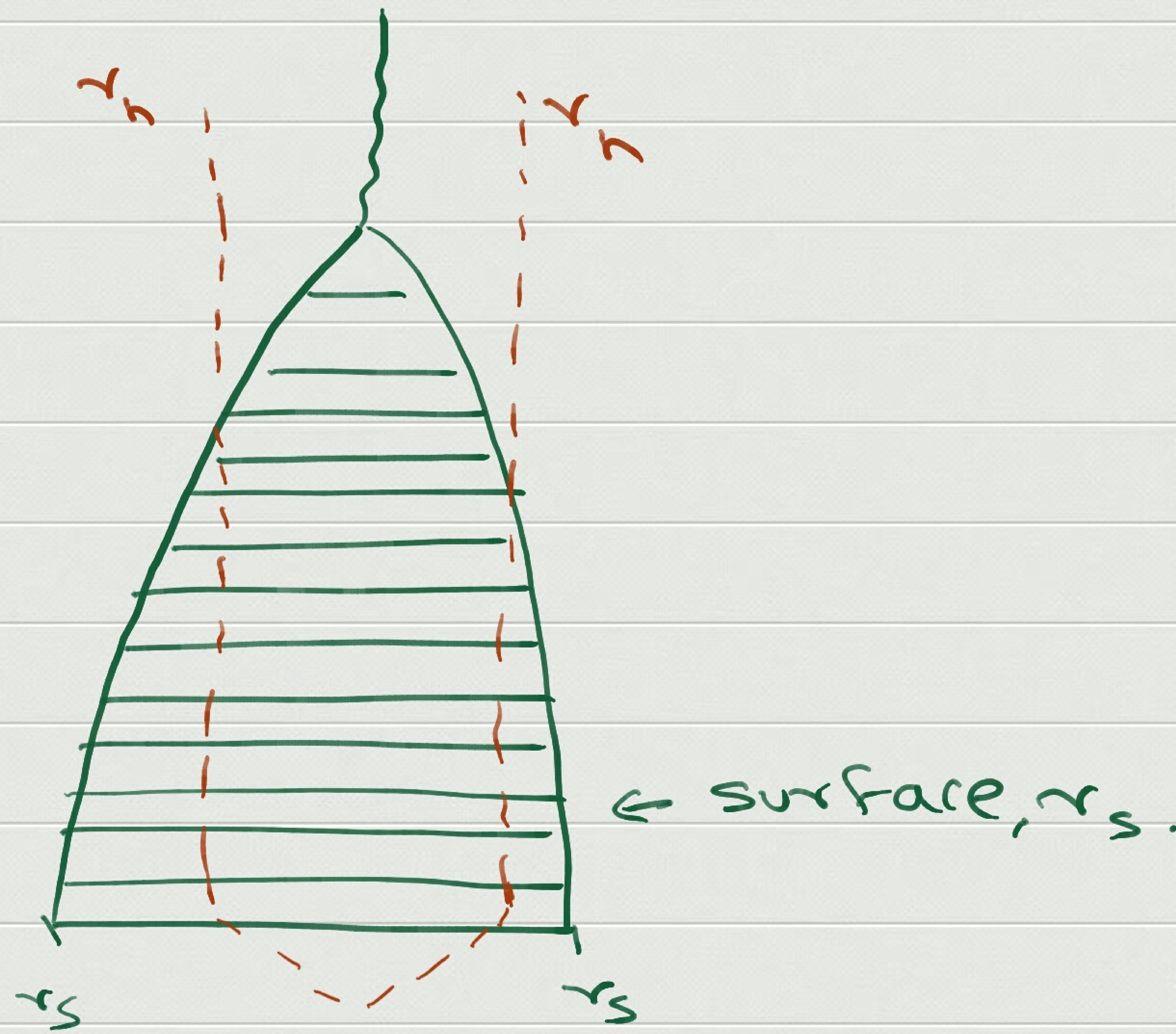
$$-\left(1-\frac{2m}{x}\right)dt^2 + \frac{dx^2}{\left(1-\frac{2m}{x}\right)} = -d\tau^2 + \frac{dr^2}{h^{2/3}}$$

the fact that $\left(\frac{dx^2}{\left(1-\frac{2m}{x}\right)} + d\tau^2 - \frac{dr^2}{h^{2/3}}\right)\frac{1}{\left(1-\frac{2m}{x}\right)}$

is the square of a differential is already enough to fit $\underline{m = \frac{4}{3}\pi r_s^3 \rho_0}$ as expected.

We are now interested in the physics of this solution. Although we have a coordinate r, one invariant way to measure it is by the radius of the S^2.
We see that this goes to zero if we sit at fixed r, in finite τ.

The way to understand this is as follows.



Another special role is played by r_n.
To see this consider trajectories of constant x.

$$dx = 0 \Rightarrow r\,h^{2/3} = \text{const.}$$

$$h(r,\tau) = \left(1 - \sqrt{6\pi\rho_0}\,\tau\,\frac{r_s^{3/2}}{r^{3/2}}\right)$$

$$\Rightarrow dr\,h^{2/3} + h^{1/3}\sqrt{6\pi\rho_0}\,\tau\,\frac{r_s^{3/2}}{r^{3/2}}\,dr - \sqrt{6\pi\rho_0}\,d\tau\,\frac{r_s^{3/2}}{r^{3/2}} = 0$$

or

$$h^{1/3}\,dr\left(h^{1/3} + \sqrt{6\pi\rho_0}\,\tau\,\frac{r_s^{3/2}}{r^{3/2}}\right) = \sqrt{6\pi\rho_0}\,\frac{r_s^{3/2}}{r^{3/2}}\,d\tau$$

At a particular value of r, it happens that

$$\left(\frac{dr}{d\tau}^2\right)_x = \frac{g_{\tau\tau}}{g_{rr}} = h^{2/3}$$

this happens at

$$x = \frac{8}{3}\pi r_s^3 \rho_0 = 2m \quad \text{(as expected)}$$

So the line of constant x is null. The causal structure of this solution is as follows