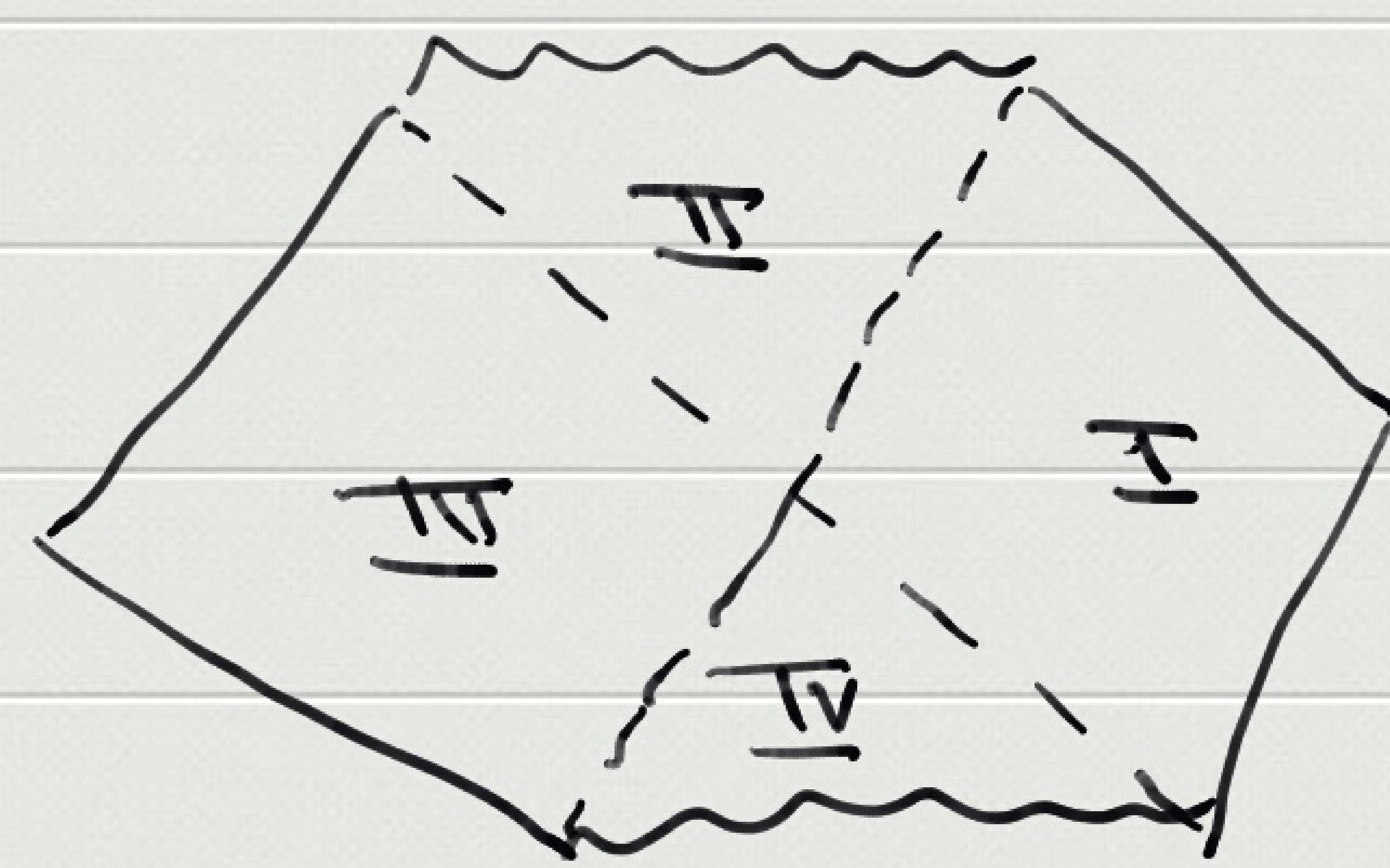
SERC SCHOOL- LECTURE 2 - 27 NOV 2015

At the end of the previous lecture, we discussed the Penrose diagram of a Schwarzschild black hole. We found that the full Penrose diagram had 4 regions!
Today, we will first discuss more realistic collapse, to show that basically only regions I & II are physical



Consider a spherical ball of dust, which is collapsing under its own gravitational interactions.

Outside the shell, we know from the theorem we proved yesterday we can write

$$ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \frac{dr^2}{\left(1-\frac{2m}{r}\right)} + r^2 d\Omega^2$$

Inside the shell, we try and pick co-moving coordinates.

$$ds^2 = d\tau^2 + A\, dR^2 + B\, d\Omega^2$$

Using our analysis yesterday you can check it is possible to put the metric in this form. We now consider a pressureless fluid, so $T^{\tau\tau} = \rho_0(\tau, R)$.

Now we find

$$G_{\tau r} = \left(A B' \dot{B} + B B' \dot{A} - 2 A B \dot{B}'\right) / \left(2 A B^2\right)$$

Use

$$\boxed{A = \frac{(B')^2}{4B}} \quad ; \quad \dot{A} = \frac{2 B' \dot{B}'}{B} - \frac{(B')^2 \dot{B}}{B^2}$$

$$A B' \dot{B} = \frac{(B')^3 \dot{B}}{B} \; ; \quad B B' \dot{A} = -\frac{(B')^3 \dot{B}}{B} + (B')^2 \dot{B}'$$

$$2 A B \dot{B}' = (B')^2 \dot{B}' \Rightarrow G_{\tau r} = 0$$

This is not the most general solution, but will do for our purpose.

With this choice, remaining components of G simplify.

$$G_{rr} = (B')^2 (\dot{B}^2 - 4 B \ddot{B}) / 16 B^3$$

so

$$G_{rr} = 0 \Rightarrow \frac{d}{d\tau}(\dot{B} B^n) = n \dot{B}^2 B^{n-1} + B^n \ddot{B}$$

here $\frac{1}{n} = -4 \Rightarrow n = -\frac{1}{4}$.

So $\frac{d}{d\tau}(\dot{B} B^{-1/4}) = 0$ and $\dot{B} = B^{1/4} f_0(r)$

or

$$\boxed{B^{3/4} = F(r)\tau + g(r).}$$

This also sets $G_{\theta\theta} = G_{\phi\phi} = 0$. So we have only the $G_{\tau\tau}$ component.

We can now specify the density at $t=0$, using

$$G_{\tau\tau} = \frac{4 f f'}{3(tf+g)(tf'+g')}$$

Now r is an arbitrary coordinate. We can use this freedom to get rid of g. The metric is

$$ds^2 = -d\tau^2 + \frac{4}{9}\frac{(\tau f'(r) + g'(r))^2}{(\tau f(r) + g(r))^{2/3}} dr^2 + (\tau f(r) + g(r))^{4/3}(d\theta^2 + \sin^2\theta\, d\phi^2).$$

at $\tau = 0$ if we choose $g(r) = r^{3/2}$, we see that on $\tau = 0$ the length element becomes

$$ds^2_{\tau=0} = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

this is not the most general solution and goes back to our simplified relation between A and B that prevents us from choosing $\dot{p}$ at $t=0$. But we can still choose p_0. We choose it to be a constant

$$
\begin{aligned}
p(\tau=0, r) &= p_0, \quad r < r_s ; \\
&= 0, \quad r > r_s.
\end{aligned}
$$

then we find that to satisfy

$$G_{\tau\tau} = 8\pi T_{\tau\tau}$$

we need

$$
\begin{aligned}
\frac{8}{9}\frac{f f'}{r^2} &= 8\pi p_0 , \quad r < r_s \\
&= 0, \quad r > r_s
\end{aligned}
$$

So

$$\frac{4}{9} f^2 = \frac{8\pi \rho_0 r^3}{3} \quad ; \quad f = -\sqrt{6\pi \rho_0 r^3}$$ [neg. sign explained below]

With this data, the metric and the evolution of the stress tensor is now fixed.

The metric is:

$$ds^2 = -d\tau^2 + (1 - \sqrt{6\pi\rho_0}\,\tau)^{4/3} dr^2 + r^2 (1 - \sqrt{6\pi\rho_0}\,\tau)^{4/3} d\Omega^2 \quad , \quad r < r_s$$

with $$T_{\tau\tau} = \frac{\rho_0}{(1 - \sqrt{6\pi\rho_0}\,\tau)^2} \quad , \quad r < r_s$$

$$ds^2 = -d\tau^2 + \frac{dr^2}{h(r,\tau)^{2/3}} + r^2 h(r,\tau)^{4/3} d\Omega^2$$

with

$$h(r,\tau) = \left(1 - \sqrt{6\pi\rho_0}\,\tau\,\frac{r_s^{3/2}}{r^{3/2}}\right)$$

and $\quad T_{\mu\nu} = 0 \quad$ outside

Since the outside solution is a vacuum solution, it can be reduced to the standard Schwarzchild form.

Tutorial question: do this transformation and determine the mass, m.

Tutorial answer:

First set $x^2 = r^2 h^{4/3}$ or $x = r h^{2/3}$

we want t so that

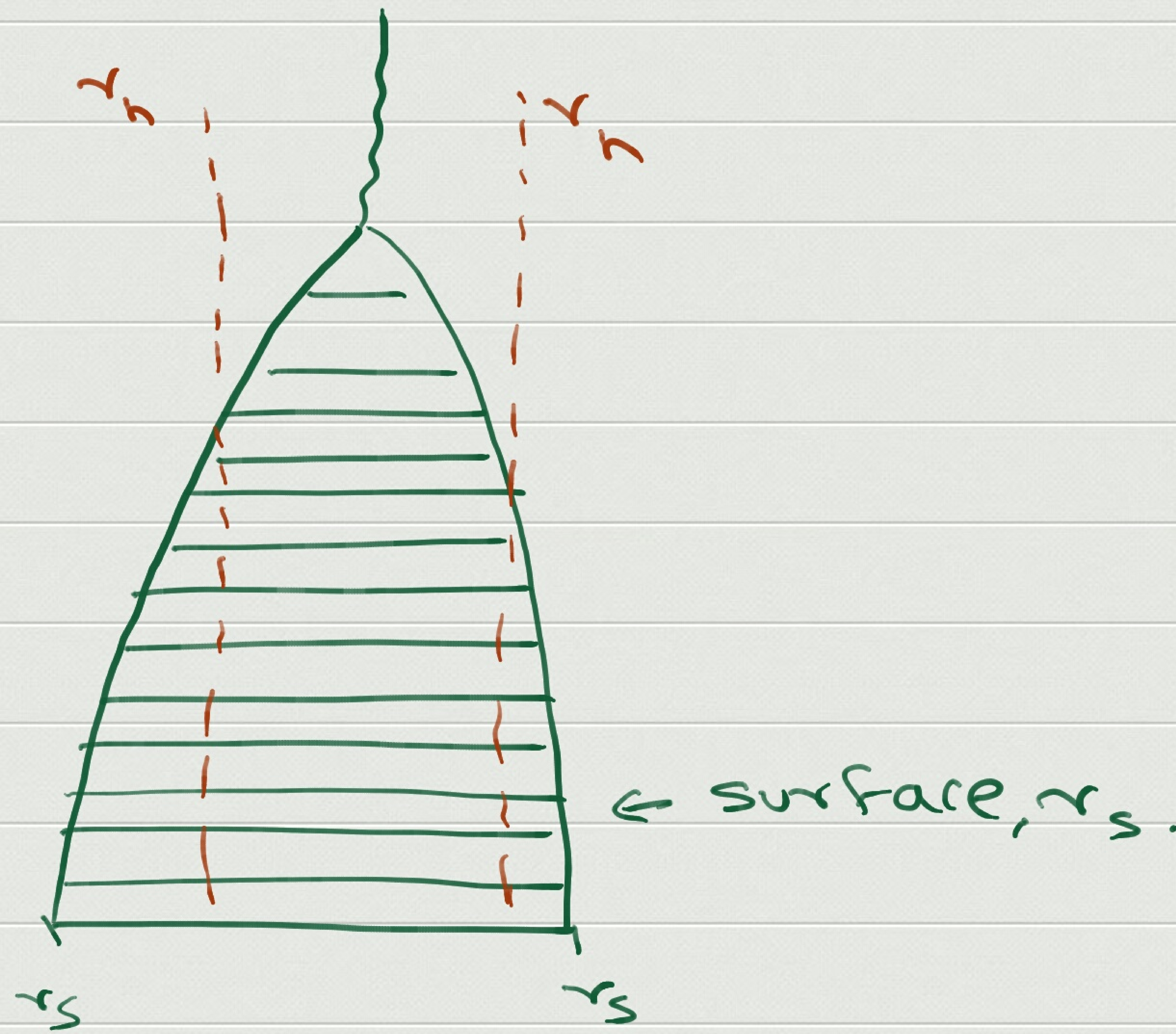
$$-\left(1-\frac{2m}{x}\right)dt^2 + \frac{dx^2}{\left(1-\frac{2m}{x}\right)} = -d\tau^2 + \frac{dr^2}{h^{2/3}}$$

the fact that $\left(\frac{dx^2}{\left(1-\frac{2m}{x}\right)} + d\tau^2 - \frac{dr^2}{h^{2/3}}\right)\frac{1}{\left(1-\frac{2m}{x}\right)}$

is the square of a differential is already enough to fit $\underline{m = \frac{4}{3}\pi r_s^3 \rho_0}$ as expected.

We are now interested in the physics of this solution. Although we have a coordinate r, one invariant way to measure it is by the radius of the S^2.
We see that this goes to zero if we sit at fixed r, in finite τ.

The way to understand this is as follows.



Another special role is played by r_h.
To see this consider trajectories of constant x.

$$dx = 0 \implies r\, h^{2/3} = \text{const.}$$

$$h(r, \tau) = \left(1 - \frac{\sqrt{6\pi\rho_0}\, \tau\, r_s^{3/2}}{r^{3/2}}\right)$$

$$\implies dr\, h^{2/3} + h^{1/3} \frac{\sqrt{6\pi\rho_0}\, \tau\, r_s^{3/2}}{r^{3/2}}\, dr - \frac{\sqrt{6\pi\rho_0}\, d\tau\, r_s^{3/2}}{r^{3/2}} = 0$$

or

$$h^{1/3}\, dr \left(h^{1/3} + \frac{\sqrt{6\pi\rho_0}\, \tau\, r_s^{3/2}}{r^{3/2}} \right) = \frac{\sqrt{6\pi\rho_0}\, r_s^{3/2}}{r^{3/2}}\, d\tau$$

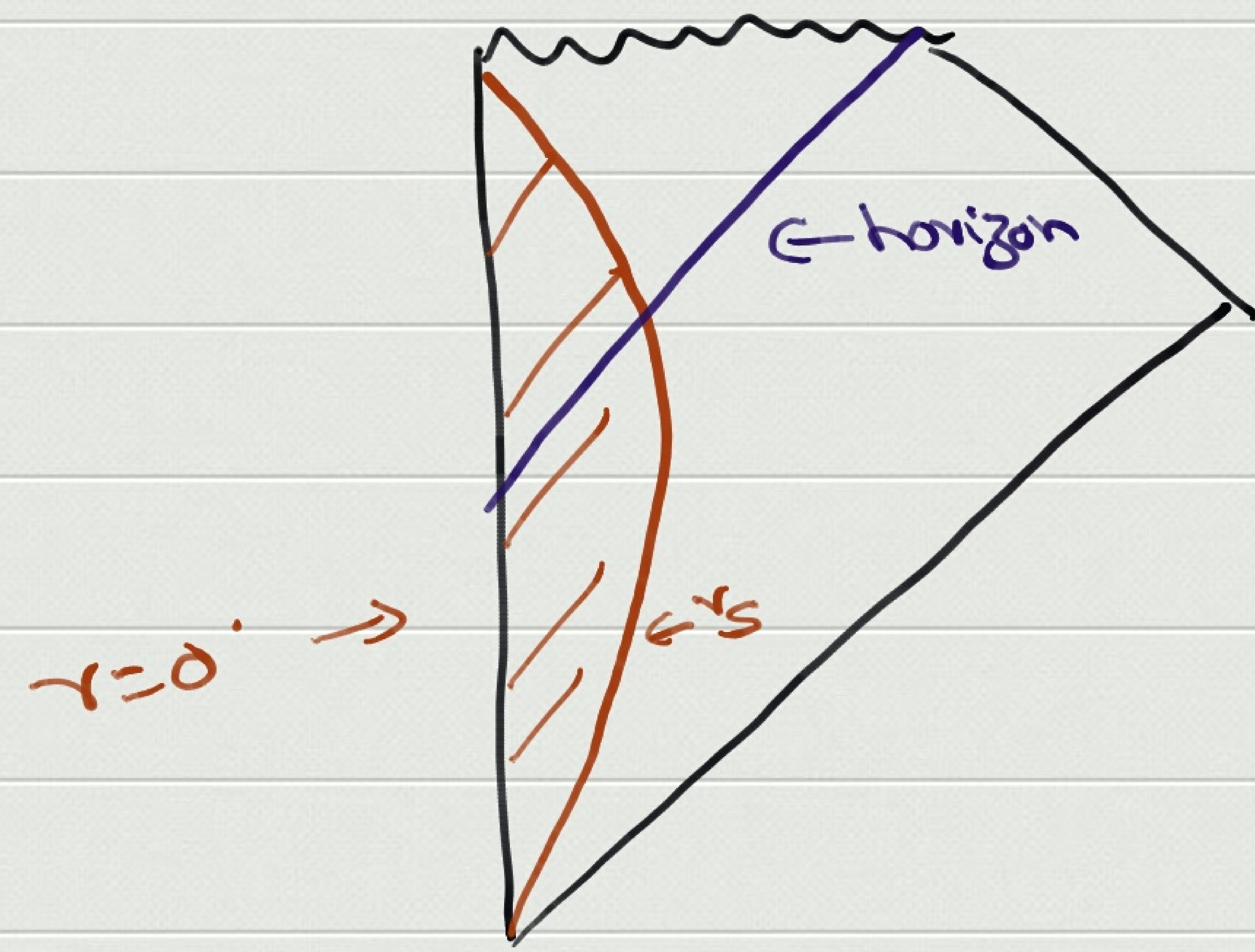
At a particular value of r, it happens that

$$\left(\frac{dr}{d\tau}\right)_x^2 = \frac{g_{\tau\tau}}{g_{rr}} = h^{2/3}$$

this happens at [Check in tutorial]

$$x = \frac{8}{3}\pi r_s^3 \rho_0 = 2m \quad \text{(as expected)}$$

So the line of constant x is null. The causal structure of this solution is as follows



We will now change gears and move on to black hole solutions with charge and angular momentum

The first one is the Reissner-Nordstrom black hole. This is the solution

$$ds^2 = -\left(1-\frac{2m}{r}+\frac{e^2}{r^2}\right)dt^2 + \frac{dr^2}{1-\frac{2m}{r}+\frac{e^2}{r^2}} + r^2 d\Omega^2$$

This satisfies the Einstein-Maxwell equations

The horizon now occurs when

$$\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right) = 0$$

or

$$r^2 - 2mr + e^2 = 0$$

this has solutions

$$r = \left(m \pm \sqrt{m^2 - e^2}\right).$$

and these are real provided $m > e$.

the larger solution is called the outer horizon whereas the smaller solution is called the inner horizon.

The R-N black hole has an extremal limit at $m = e$. For charge values larger than this the solution has a naked singularity which is problematic.

So it is believed that black holes beyond this bound do not exist.

Tutorial question: restore factors of G, c etc. and compute $\frac{q}{m}$ for an electron and a proton.

We can also add angular momentum to the black hole. This results in the Kerr-Newman black hole.

$$ds^2 = -\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)dt^2 - 2a\sin^2\theta\frac{(r^2+a^2-\Delta)}{\Sigma}\,dt\,d\phi$$
$$+\left[\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right]\sin^2\theta\,d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma\,d\theta^2$$

Here $\Sigma = r^2 + a^2\cos^2\theta$; $\Delta = r^2 + a^2 + e^2 - 2Mr$

The astrophysical significance of kerr black-holes is that black-hole accretion disks tend to spin up black holes. On the other hand charge tends to get neutralized. So, most astrophysical black holes are believed to be kerr black holes.

The kerr black hole has many interesting properties, some of which we will return to next time.

Note, for one, that while it has a timelike killing vector, $\frac{\partial}{\partial t}$ is not hypersurface orthogonal.

So, the Kerr metric is stationary but not static.

There is an extremal limite

$$M^2 = a^2 + e^2$$

The angular momentum is $J = Ma$ [again defined from asymptotics.]