SERC SCHOOL, LECTURE 3, 28 NOV 2015

No-Hair Theorem : The no-hair theorem states that the Kerr-Newman black hole is the most general stationary black hole solution that is asymptotically flat.

We might have thought that one could generate solutions by having some complicated motion. However if this motion is confined to a compact region, there must be acceleration, and grav. radiation and so the solution will collapse.

This situation appears similar to e.m. where one cannot get stable solutions with q.m. and in the classical theory things collapse or radiate their energy away.

What is special about G.R. is that we can find smooth stable solutions that have a lot of energy but still have no hair.
Of course, these solutions are smooth only if we look from outside the horizon.

If we are willing to change the asymptotic structure we can find more general solutions.
However to change asymptotics, we need som energy density that extends out to infinity. One way to do this is to use a cosmological constant.
In the presence of such a term, the eom become

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$$

The vacuum solution with $\Lambda > 0$ is called de Sitter space. The solution with $\Lambda < 0$ is called anti-de Sitter space.

We are interested in anti-de Sitter space because that allows us to build models of Q.G.
The generalization of the Schwarzschild solution to AdS is

$$f(r) = \frac{r^2}{l^2} + 1 - \frac{2m}{r}.$$

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2\,d\Omega^2$$

with $\Lambda = -3l^2$

We now want to investigate the causal structure of these solutions. We will consider the R-N metric and the AdS-Schwarzschild metric.

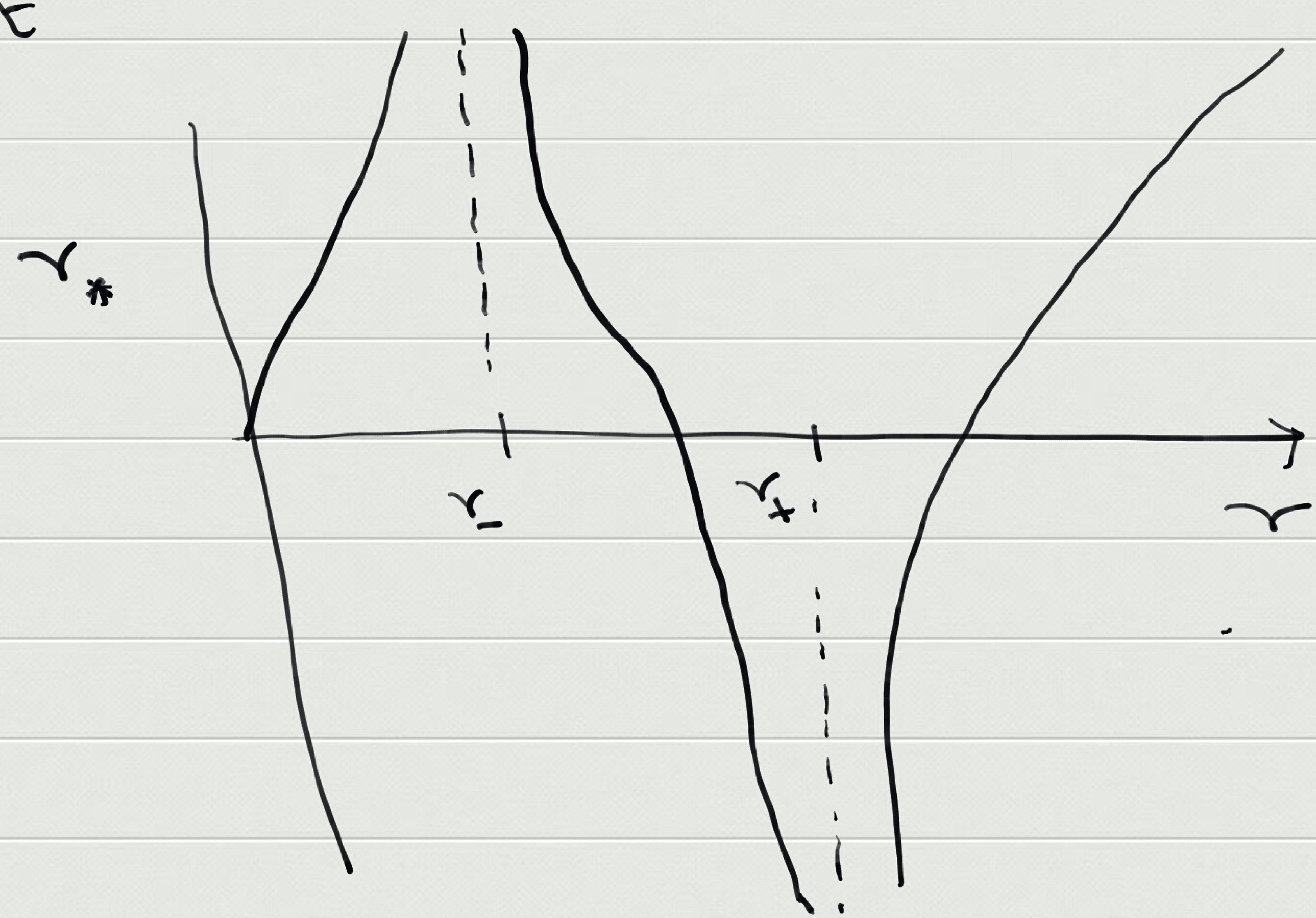
Recall that we have

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)} + r^2 d\Omega^2$$

We again introduce a tortoise coordinate through

$$dr_*^2 = \frac{dr^2}{\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^2} \Rightarrow r_* = r + \frac{1}{r_+ - r_-}\left[r_+^2 \ln\left|\frac{r - r_+}{r_+}\right| - r_-^2 \ln\left|\frac{r - r_-}{r_-}\right|\right]$$

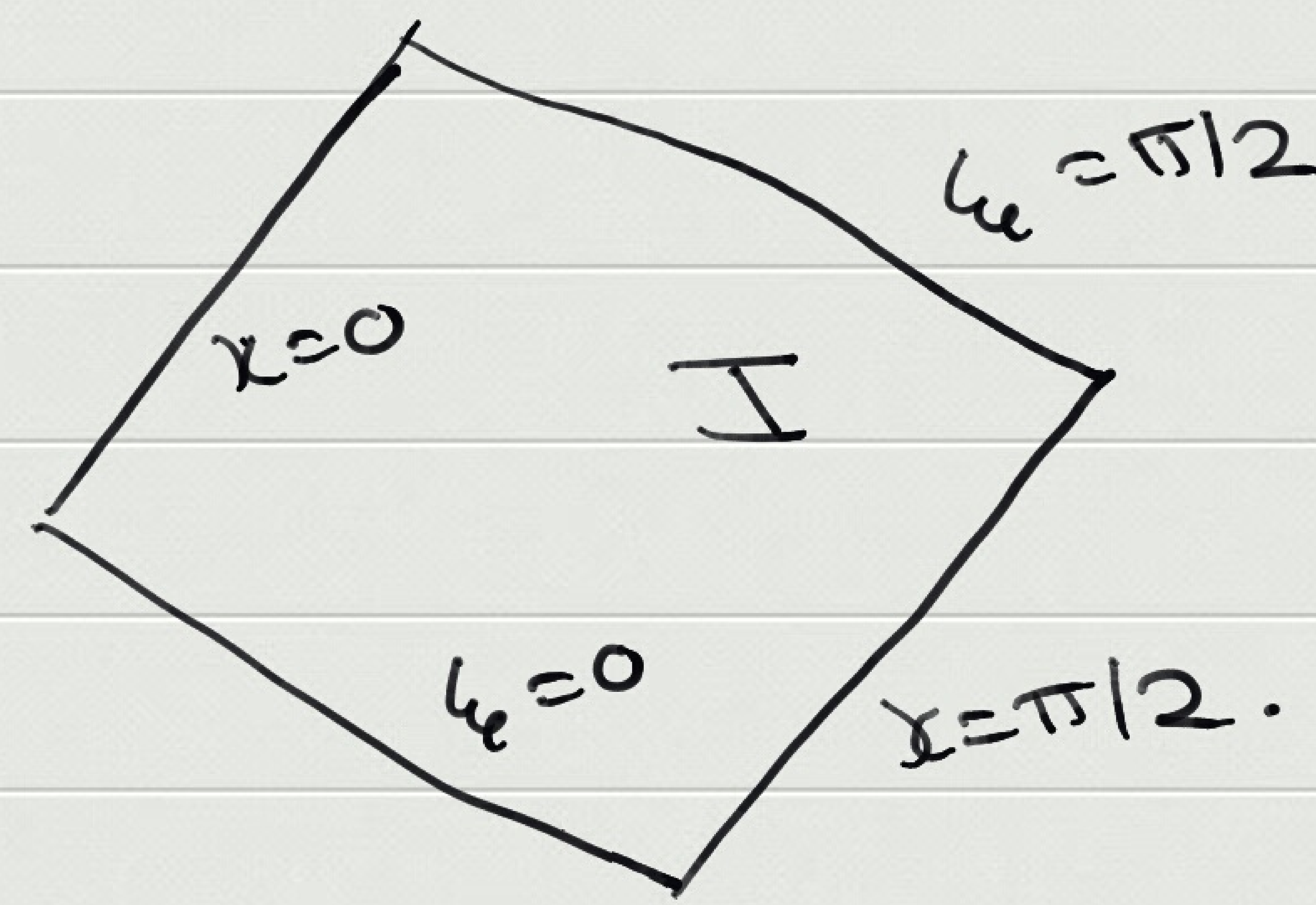
Now the graph of r_* vs r looks somewhat different



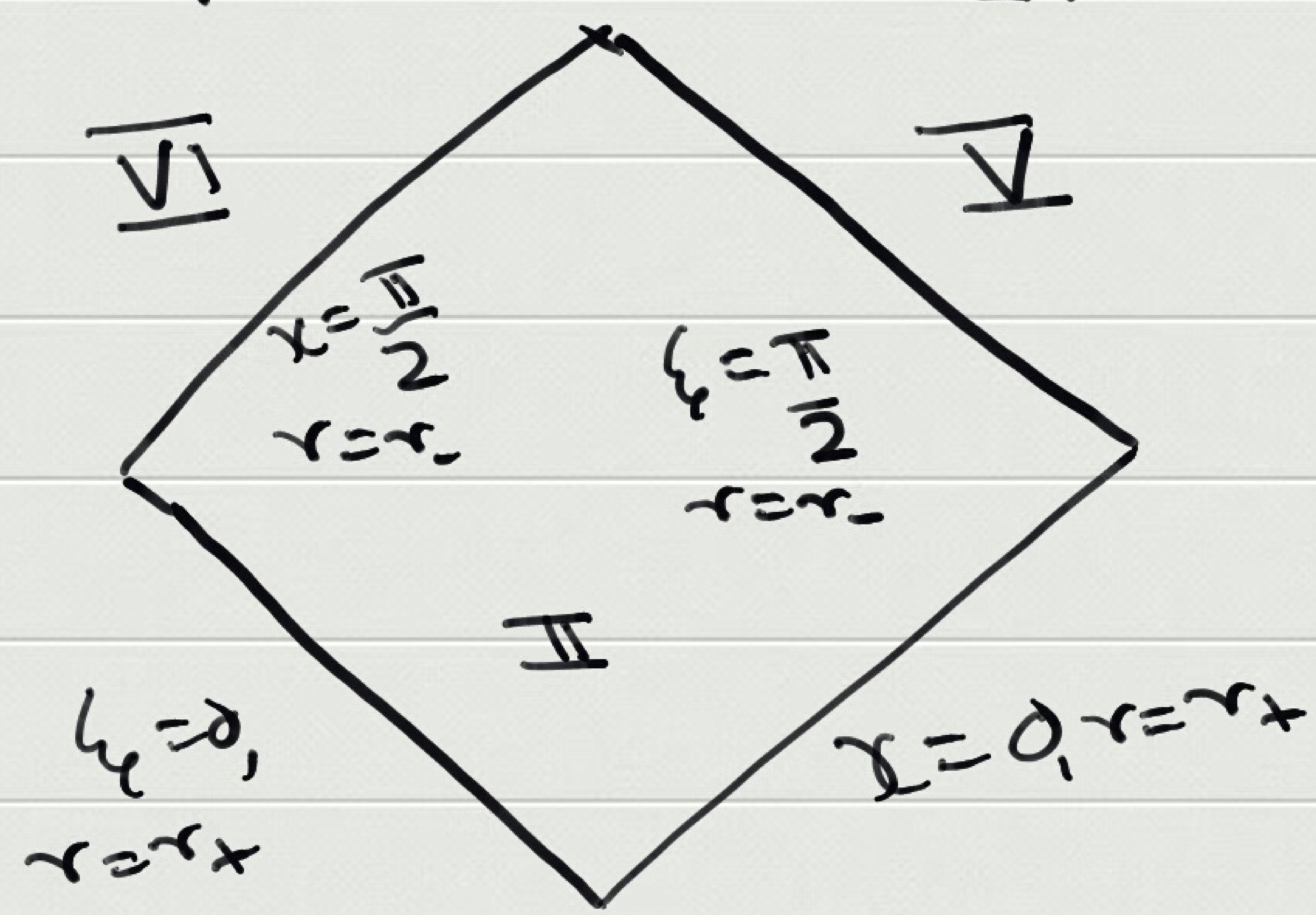
corresponding to this, let us introduce coordinates

$$U = -e^{\alpha(r_* - t)}, \quad V = e^{(r_* + t)}, \quad \chi = \tan^{-1} U, \quad \xi = \tan^{-1} V$$

as usual we see that $r_* \to -\infty$ corresponds to $UV = 0$
So the diagram for region I covered vy the original patch is

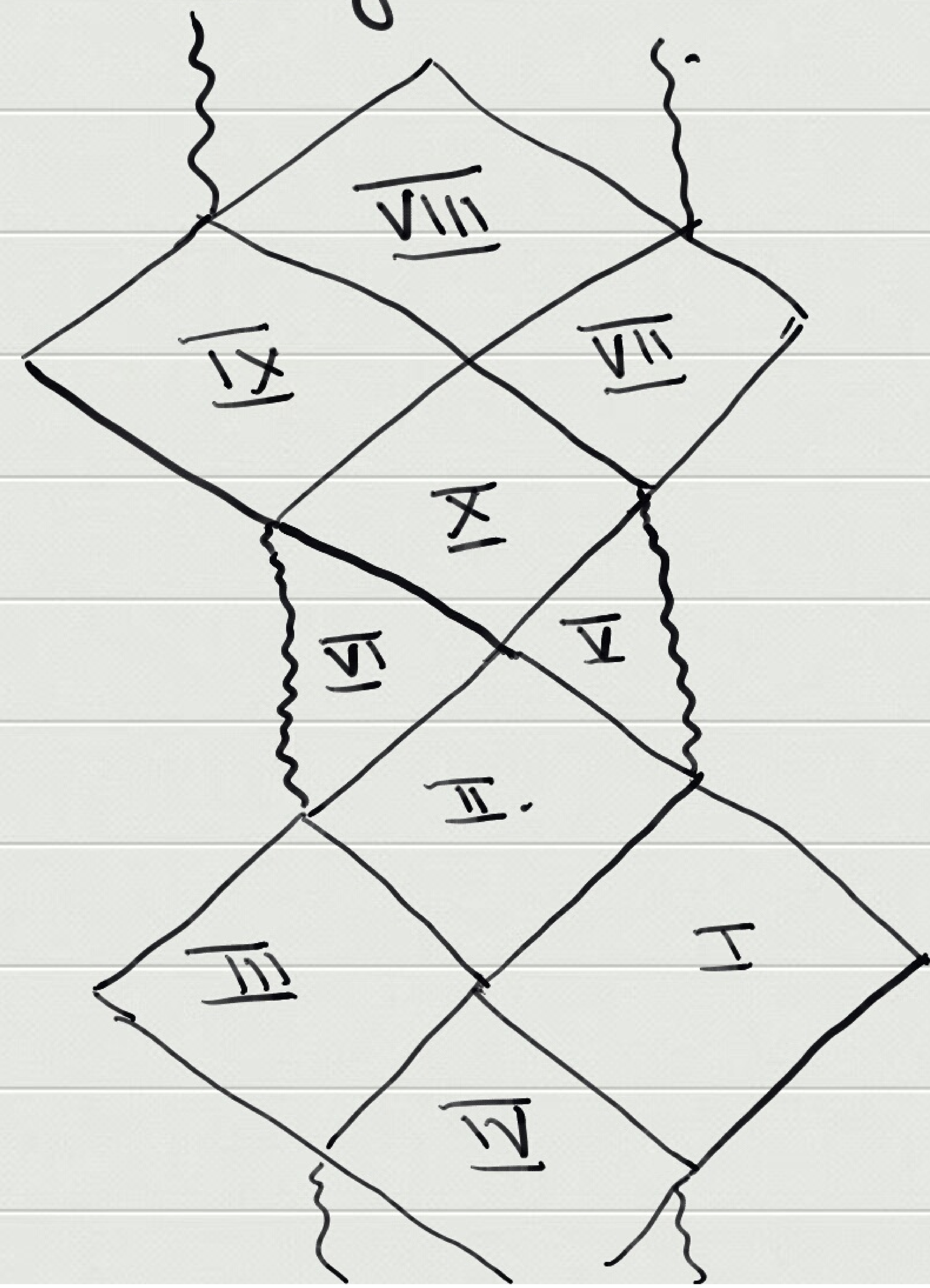


Now we extend past $V=0$, As usual lines of constant r_* are hyperboloids. But now note that $UV=1$, is not a singular point, because this corresponds to some value of r intermediate between r_+ and r_-. We can now go past $\xi = \frac{\pi}{2}$, and $\chi = \frac{\pi}{2}$ corresponding to the part of $r < r_-$ into regions $\underline{\overline{V}}$ and $\underline{\overline{VI}}$.



in regions $\overline{\underline{V}}$ and $\overline{\underline{VI}}$, we eventually hit the singularity

So the full diagram looks like

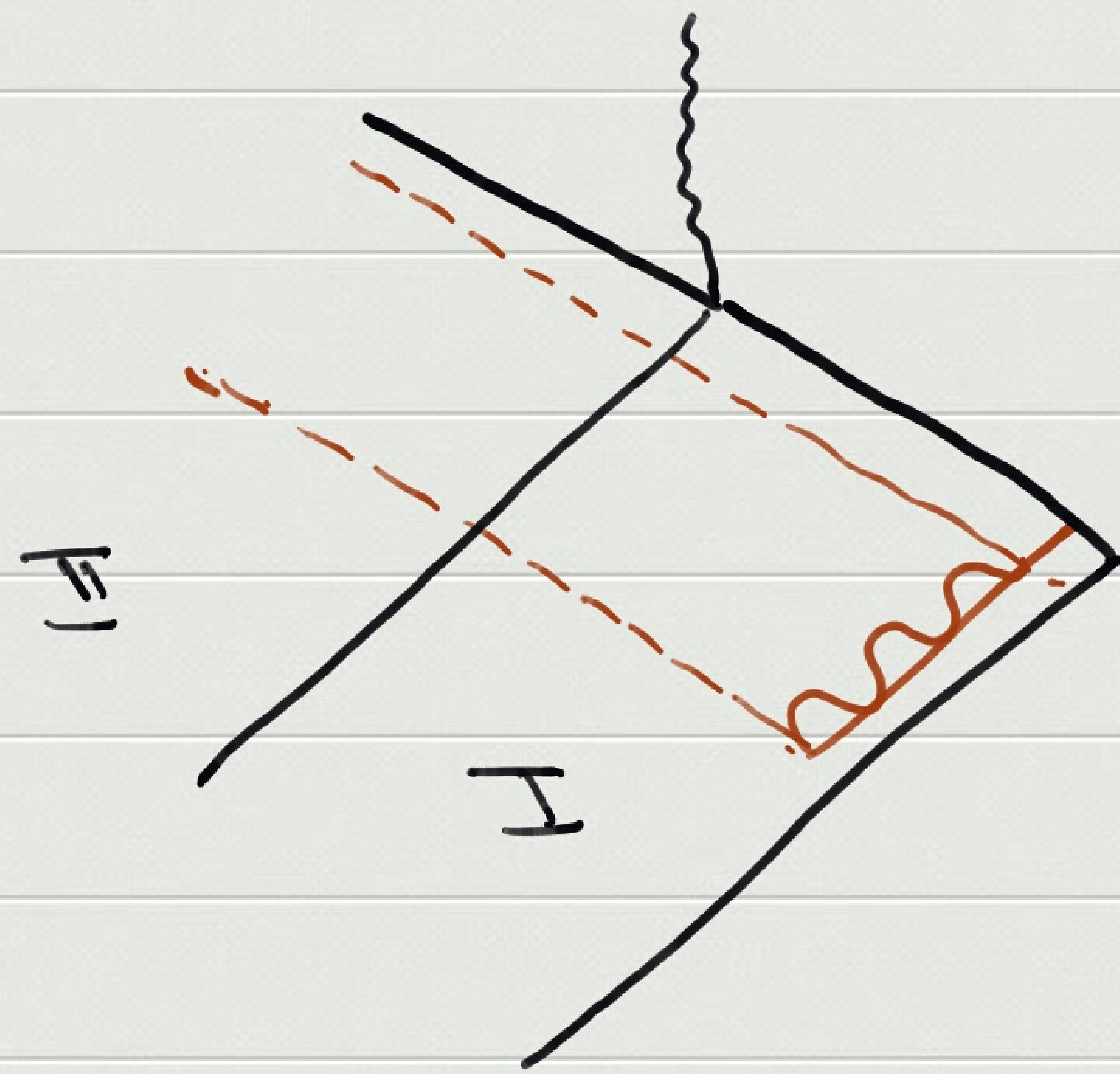


however this diagram is also somewhat fake.
There are 2 reasons:

a) as we saw in the Oppenheimer-Snyder collapse, a collapsing star covers most of the "left" of the diagram.

b) the inner horizon is actually unstable to small perturbations.

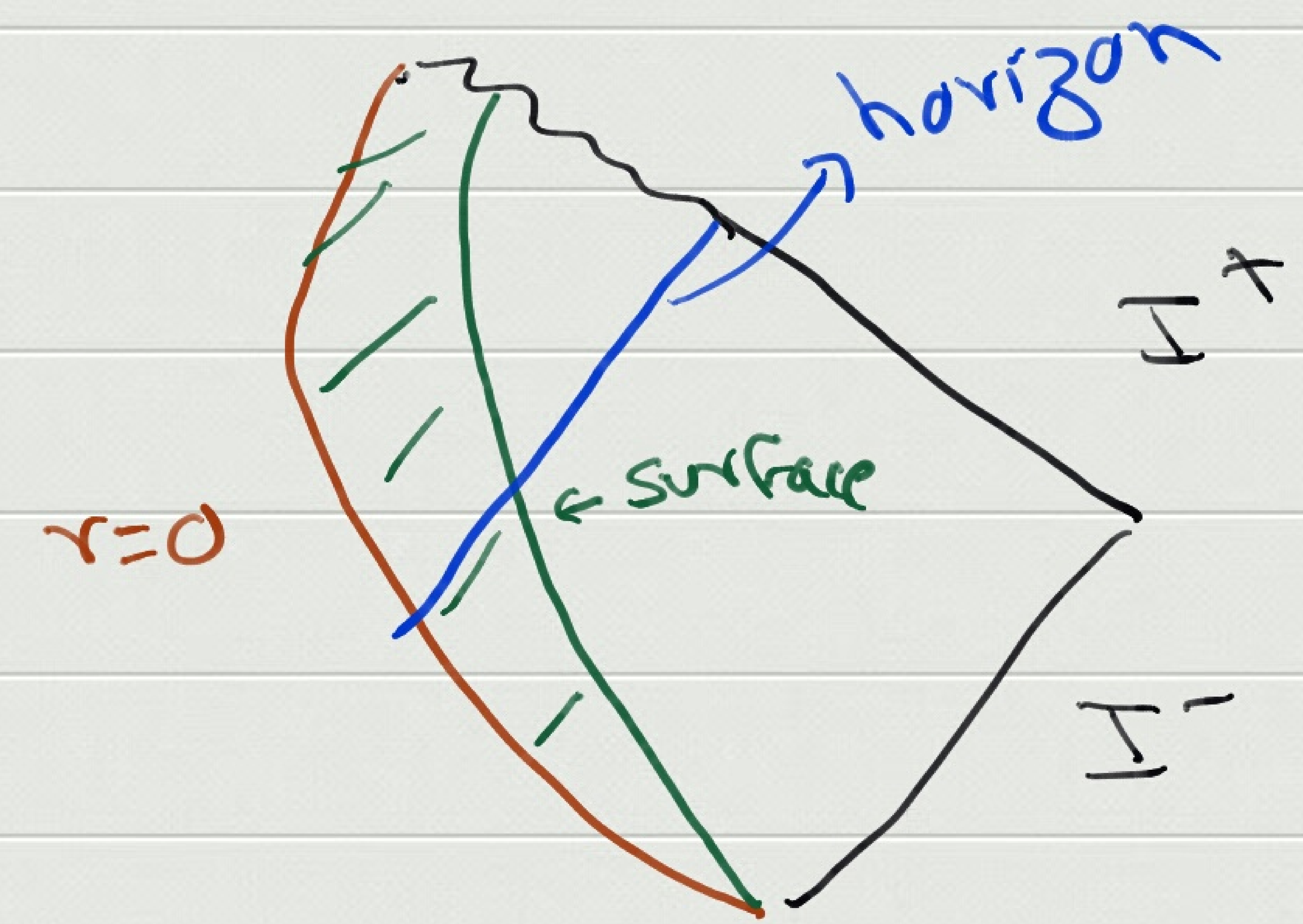
To see this, consider specifying some initial data at past null infinity



because there is an infinite affine distance from $\zeta_e = 0$ to $\zeta_e = \pi/2$ if we think of a plane wave solution it is fine if it has infinite oscillations in this interval.

But an observer inside the horizon reaches from r_+ to r_- in finite proper time. So he sees all these infinite oscillations in finite proper time $\longrightarrow$ infinite blue shift!

So a more realistic diagram is

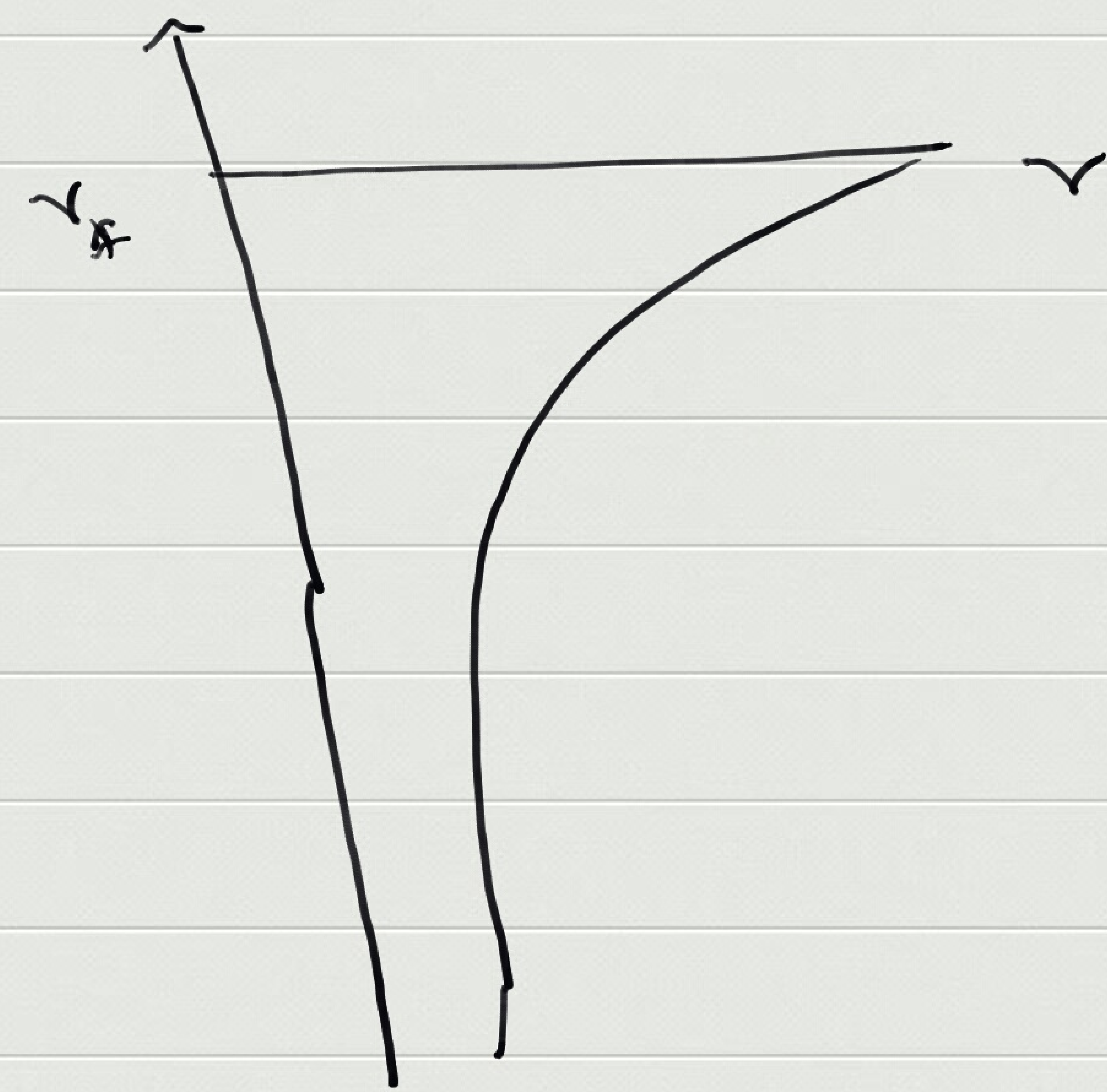


Now we turn to the causal structure of the AdS-Schwarzschild black hole. Once again we introduce a tortoise coordinate through

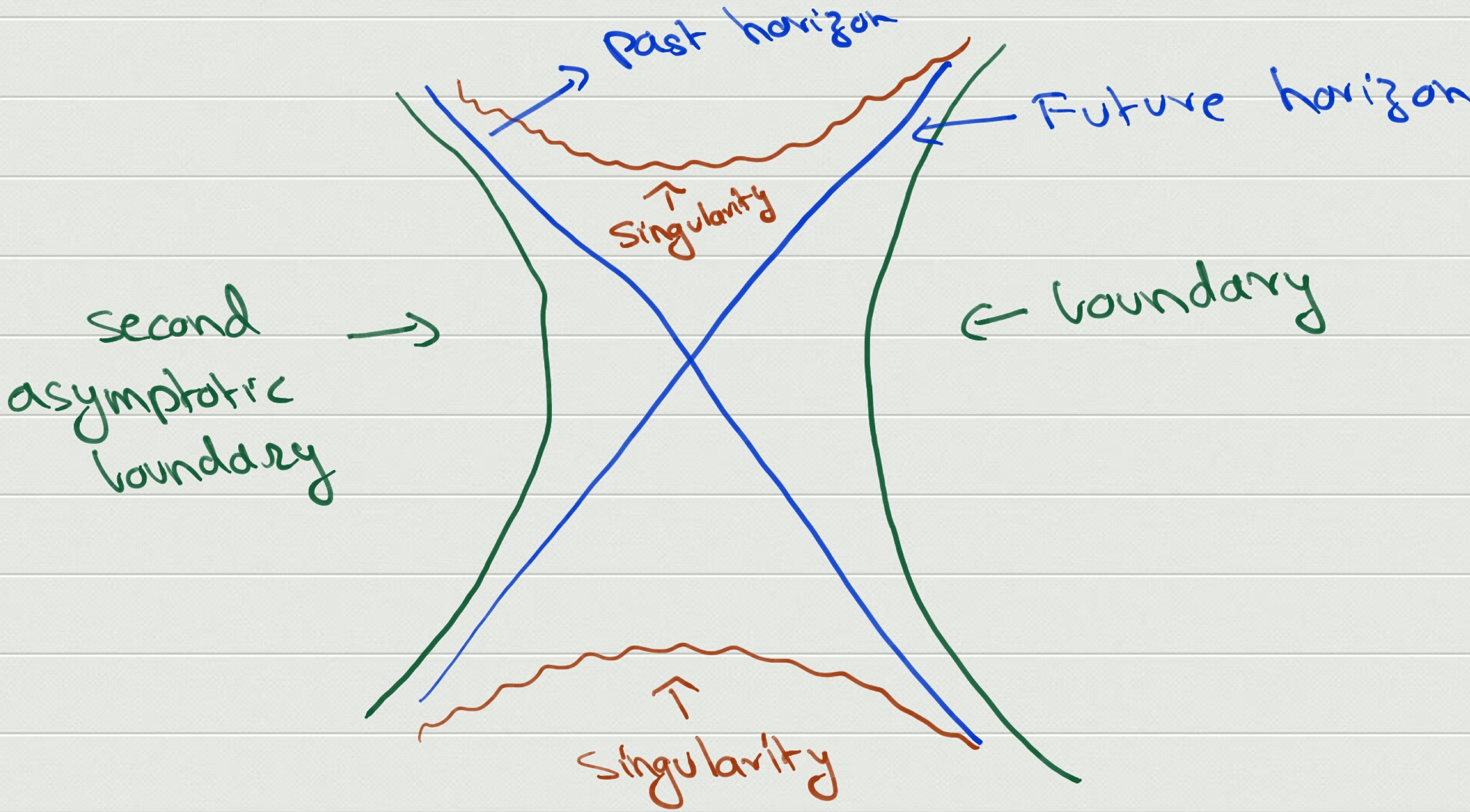
$$\frac{dr_*}{dr} = \frac{1}{r^2+1-\frac{2m}{r}}$$

Now, what is new here is that the integral over r is convergent as $r \to \infty$. Near $r \to r_h$ as usual we have $r_* \to -\infty$. At $r \to \infty$, we can set $r_* = 0$

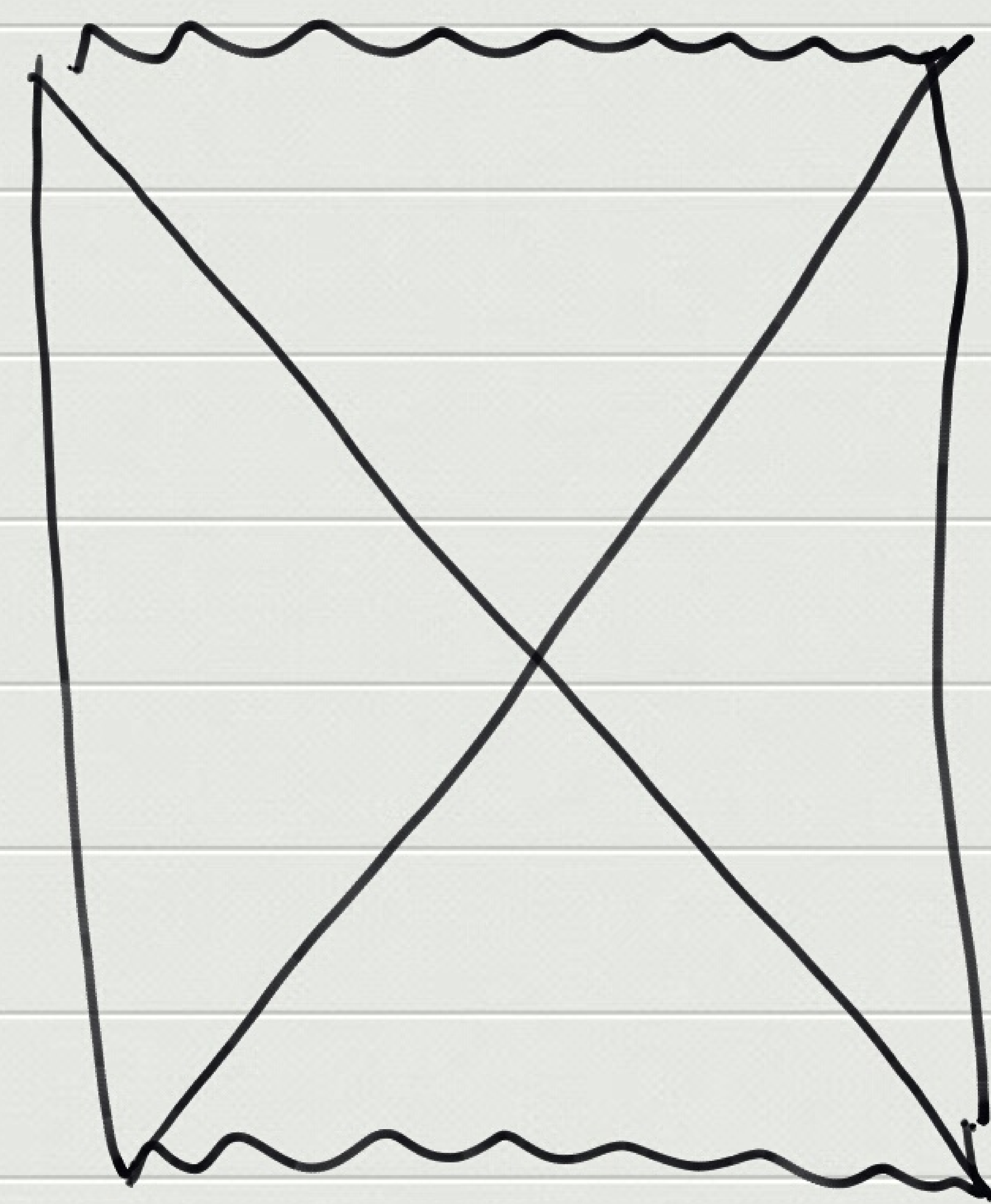
So the $r-r_*$ graph now looks as follows



Again introducing null coordinates, U, V, we find


past horizon
Future horizon
singularity
second
asymptotic
boundary
boundary
singularity

You may often see a Penrose diagram which looks as follows



the novel feature here is the prescence of a boundary.
So this is like a black hole in a box

Now we return to the Kerr Black Hole. Its causal structure is similar to the R-N B.H. But it has some other interesting properties

The first is that the horizon is not where $g_{tt}=0$ but rather where $\Delta=0$ or

$$r^2+a^2+e^2-2Mr=0$$

Consider $\chi=\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\phi}$.

with $\Omega=\frac{a}{r_+^2+a^2}$

Tutorial: Show χ becomes null at the horizon.

Solution:

Its norm is

$$g_{\mu\nu} \chi^\mu \chi^\nu = -\frac{(\Delta - a^2 \sin^2\theta)}{\Sigma} - \frac{2a\sin^2\theta\,(r^2+a^2-\Delta)\,\Omega}{\Sigma}$$
$$+ \left(\frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma} \right) \sin^2\theta\,\Omega^2$$

$$= -\frac{1}{\Sigma} \Big[\Delta(1 - 2a\sin^2\theta\,\Omega + a^2\sin^4\theta\,\Omega^2) - a^2\sin^2\theta$$
$$+ 2a\sin^2\theta\,(r^2+a^2)\Omega$$
$$- (r^2+a^2)^2 \sin^2\theta\,\Omega^2 \Big]$$

$$= -\frac{1}{\Sigma} \left\{ \Delta\,(\#) - \sin^2\theta \left[(r^2+a^2)\Omega - a \right]^2 \right\}$$

So set

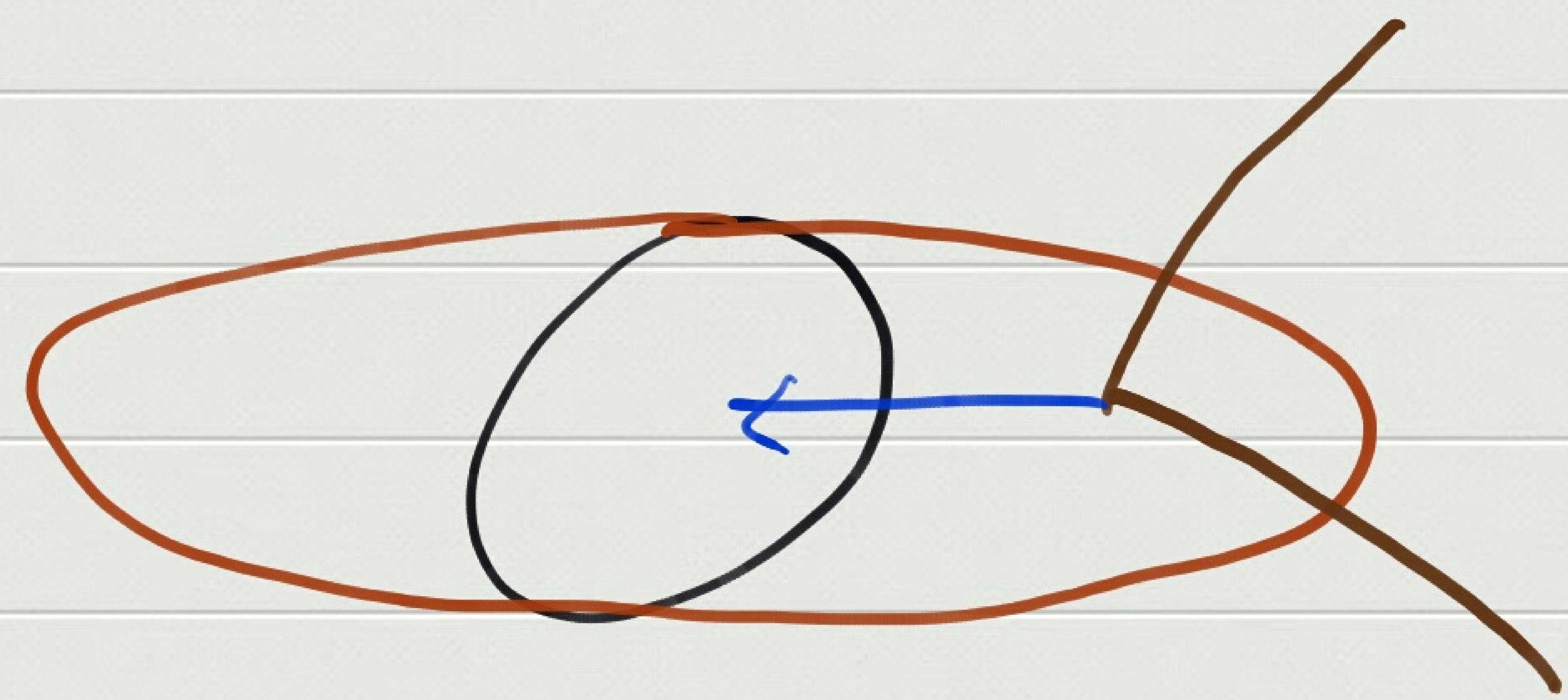
$$\Omega = \frac{a}{r_+^2 + a^2}$$

leads to the killing vector that vanishes at the outer horizon.

Second note that there is a region outside the horizon where $\frac{\partial}{\partial t}$ becomes spacelike. This is called the ergosphere

The existence of the ergosphere allows the Penrose process to extract energy from the black hole!

The idea is as follows. Consider a particle that enters the ergosphere, breaks into two parts and one part exits the ergosphere



We have two Killing vectors $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi}$ and

consequently the particle's momentum in these directions is conserved. So

note minus sign to make E +ve

$$-m_0 \frac{dx^\mu}{dt} K^\nu g_{\mu\nu} = E \text{ (energy)}$$

and $$m_0 \frac{dx^\mu}{dt} R^\nu g_{\mu\nu} = L \text{ (angular momentum)}$$

$$K = \frac{\partial}{\partial t} \quad ; \quad R = \frac{\partial}{\partial \phi}$$

Now when the particle breaks into two, we have

$$E_{in}^{(1)} = E^{(1)} + E^{(2)}$$
$$L_{in}^{(1)} = L^{(1)} + L^{(2)}$$

but notice that for particle 2 can have locally positive energy, even with

$$E^{(2)} < 0$$

so we end up with $E^{(1)} > E_{in}^{(1)}$.

However, for particle 2 to have locally +ve energy we must at least have

$$-\frac{dx_2^\mu}{d\tau}\left(k^\nu + \Omega_H R^\nu\right) g_{\mu\nu} > 0$$

and therefore we see that

$$E^{(2)} - \Omega_H L^{(2)} \geq 0$$

[relative minus sign because of the definitions of E, L]

Conversely

$$L^{(2)} \leq \frac{E^{(2)}}{\Omega_H}$$

[Note: no absolute values here.]

in any process or $\delta L \leq \frac{\delta M}{\Omega_H}$ for the B.H.