Lecture 6, SERC SCHOOL, 2 DEC 2015

At the end of our last lecture, we had two expressions

$$\left.\Phi = \int \frac{dw}{\sqrt{w}} \left[\tilde{a}_w U_L(u_r) + \tilde{b}_w V_L(V_R) + a_w U_R(u_R) + b_w V_R(V_R) + h.c \right] \right\} \text{Rindler Expansion}$$

$$\Phi = \int \frac{dw}{\sqrt{w}} \left[c_w e^{-iw(T-X)} + d_w e^{-iw(T+X)} + h.c. \right]$$

ordinary Minkowski coordinates

Finding the full Bogoliubov coeffs is hard but we can find the relationship between the vacua using a trick due to Unruh.

Consider the mode

$$U_v(u_R) = U_L^*(u_R) \,, \quad \text{in } \underline{\text{III}}$$

$$= e^{+\pi\omega/a} U_R(u_R) \,, \quad \text{in } \underline{\text{I}}$$

This can be written as

$$U_v(u_R) = U^{+i\omega/a}$$

ordinary Minkowski

$U = T - X$

[Imp: recall the minus sign $U \propto e^{-u_R}$]

But in writing this mode, we have to choose the branch cut, which we chosen in the upper-half plane.

Therefore as we continue U from +ve to -ve, we get

$$(-U)^{+i\omega/a} = \left[e^{-\pi i} |U|\right]^{+i\omega/a} = e^{+\pi \omega/a} |U|^{-i\omega/a}$$

The choice of the branch cut controls whether we get $e^{-\pi \omega/a}$ or $e^{\pi \omega/a}$

These modes have the property that

$$\int U^{+i\omega/a} e^{-i\omega' T} dT = 0 , \quad \forall \omega' > 0$$

because the integral can be continued analytically in the lower half plane.
So, we can write

$$U^{+i\omega/a} = \int_0^\infty x(\nu) e^{-i\nu T} d\omega$$

for some $x(\nu)$

The Unruh mode has only positive Minkowski frequencies.

We can also consider

$$\tilde{U}_U(u_R) = \begin{cases} U_L(u_R), & \underline{\overline{III}} \\ e^{-\pi w/a} U_R^*(u_R), & \underline{\overline{I}} \end{cases}$$

where again the branch cut was chosen in the upper half plane. and this leads to the +ve sign in the exponent.

Now we can quantize the field as

$$\phi = \int \frac{d\omega}{\sqrt{\omega}} \Big[e_\omega U_u(u_R) + \tilde{e}_\omega \tilde{U}_v(u_R)$$

$$+ f_\omega V_v(v_R) + \tilde{f}_\omega \tilde{V}_v(v_R) + h.c. \Big]$$

Tutorial: define V_v and $\tilde{V}_v$

We clearly have,

$$a_\omega = e^{+\pi\omega/a} e_\omega + e^{-\pi\omega/a} \tilde{e}_\omega^{+}$$

$$\tilde{a}_\omega = e_\omega^{+} + \tilde{e}_\omega$$

$$\tilde{a}_\omega^+ = e_\omega + \tilde{e}_\omega^+$$

so

$$e_\omega = \left(a_\omega - e^{-\pi\omega/a}\tilde{a}_\omega^+\right) / \left[e^{\pi\omega/a} - e^{-\pi\omega/a}\right]$$

Similarly

$$\tilde{e}_\omega = \left(\tilde{a}_\omega - e^{-\pi\omega/a} a_\omega^+\right) / \left[1 - e^{-2\pi\omega/a}\right]$$

[Note the asymmetries in the denominator come because we have not normalized e_ω, $\tilde{e}_\omega$.]

The Minkowski vacuum satisfies

$$e_\omega |\Omega_M\rangle = \tilde{e}_\omega |\Omega_M\rangle = 0$$

and so,

$$\left(a_\omega - e^{-\pi\omega/a}\tilde{a}_\omega^\dagger\right)|\Omega_M\rangle = 0$$

$$\left(\tilde{a}_\omega - e^{-\pi\omega/a}a_\omega^\dagger\right)|\Omega_M\rangle = 0$$

This can be written as

$$|\Omega_M\rangle = e^{\int d\omega\, e^{-\pi\omega/a} a_\omega^\dagger \tilde{a}_\omega^\dagger + b\text{-terms}} |\Omega_{I,III}\rangle$$

where

$$a_\omega |\Omega_{I,III}\rangle = \tilde{a}_\omega |\Omega_{I,III}\rangle = 0.$$

Tutorial

Write this state as

$$|\Omega_M\rangle = e^{-\beta E/2} |E_L, E_R\rangle$$

where $E_L = \sum \omega a_\omega^\dagger a_\omega \,;\, E_R = \sum \omega \tilde{a}_\omega^\dagger \tilde{a}_\omega$

Show $\beta = 2\pi/a$

Now, tracing out region $\overline{III}$, we find that the modes in 1 are in a mixed state with density matrix

$$\rho = \frac{1}{2} e^{-\beta H}$$

and $\beta = \frac{2\pi}{a}$.

$\uparrow$

CENTRAL RESULT!

To complete the story, we also write down the expansion in the "Forward" region, region II.

The "Unruh" expansion can easily be extended into region II.

Now we find that

$$e_\omega U_\nu(u_R) + \tilde{e}_\omega \tilde{U}_\nu(u_R) + e_\omega^+ U_\nu^*(u_R) + \tilde{e}_\omega^+ \tilde{U}_\nu^*(u_R)$$

$$= (e_\omega + \tilde{e}_\omega^+)\, U^{i\omega/a} + (\tilde{e}_\omega + e_\omega^+)\, U^{-i\omega/a}$$

$$= \tilde{a}_\omega^+ U^{i\omega/a} + \tilde{a}_\omega U^{-i\omega/a}$$

Tutorial

Show the V_ν and $\tilde{V}_\nu$ terms add to $b_\omega V^{-i\omega/a} + b_\omega^+ V^{i\omega/a}$.

In region II

$$U = \frac{1}{a} e^{-au_R} \quad ; \quad V = \frac{1}{a} e^{av_R}$$

and so

$$dU\,dV = \frac{1}{a^2} e^{a(v_R - u_R)} \quad (-\,du_R\,dv_R)$$

$$= \frac{1}{a^2} e^{2ax_R} (dx_R^2 - dt_R^2)$$

so x_R is the timelike coordinate and the expansion is

$$\phi_{II} = \int \frac{d\omega}{\sqrt{\omega}} \left[e^{-i\omega v_R} b_\omega + e^{i\omega v_R} \tilde{a}_\omega + h.c \right]$$

signs correspond to $-i\omega x_R$
since x_R is the time coordinate here

The result to remember is

In region II, by continuity notice that the "v" modes from the right [left movers] and the "u" modes from the left [right movers] cross over.

The right-movers from the right stay there and the left movers from the left also do not cross over.

Hawking Radiation

Now we switch gears to a consideration of Hawking radiation.

The simplest derivation of Hawking radiation goes as follows.

Consider the wave equation in the B.H. background.

In r_*, t coordinates, for s-waves this is just

$$\left(\frac{\partial}{\partial t^2} - \frac{\partial}{\partial r_*^2}\right)\Phi = 0$$

or

$$\phi = \int d\omega \left[\frac{a_\omega}{\sqrt{\omega}} e^{-i\omega(t+r_*)} + \frac{b_\omega}{\sqrt{\omega}} e^{-i\omega(t-r_*)} + h.c. \right]$$

Now we assume that an infalling observer sees the vacuum as is predicted from the equivalence principle. This observer uses the Kruskal U, V coordinates that are flat near the horizon. But locally the transformation between the U, V, t, r_* is just

$$U = -e^{(t-r_*)/4M} \quad ; \quad V = e^{(t+r_*)/4M}$$

This suggests that the Bogoliubov coefficients should be the same. So the infalling vacuum can be written as a density matrix for the b-modes with temperature

$$T = \frac{1}{8\pi M}$$

There is a slightly subtle point here. Since we assumed only that the future horizon is smooth only the b-modes, not the a-modes are populated

[More tomorrow.]