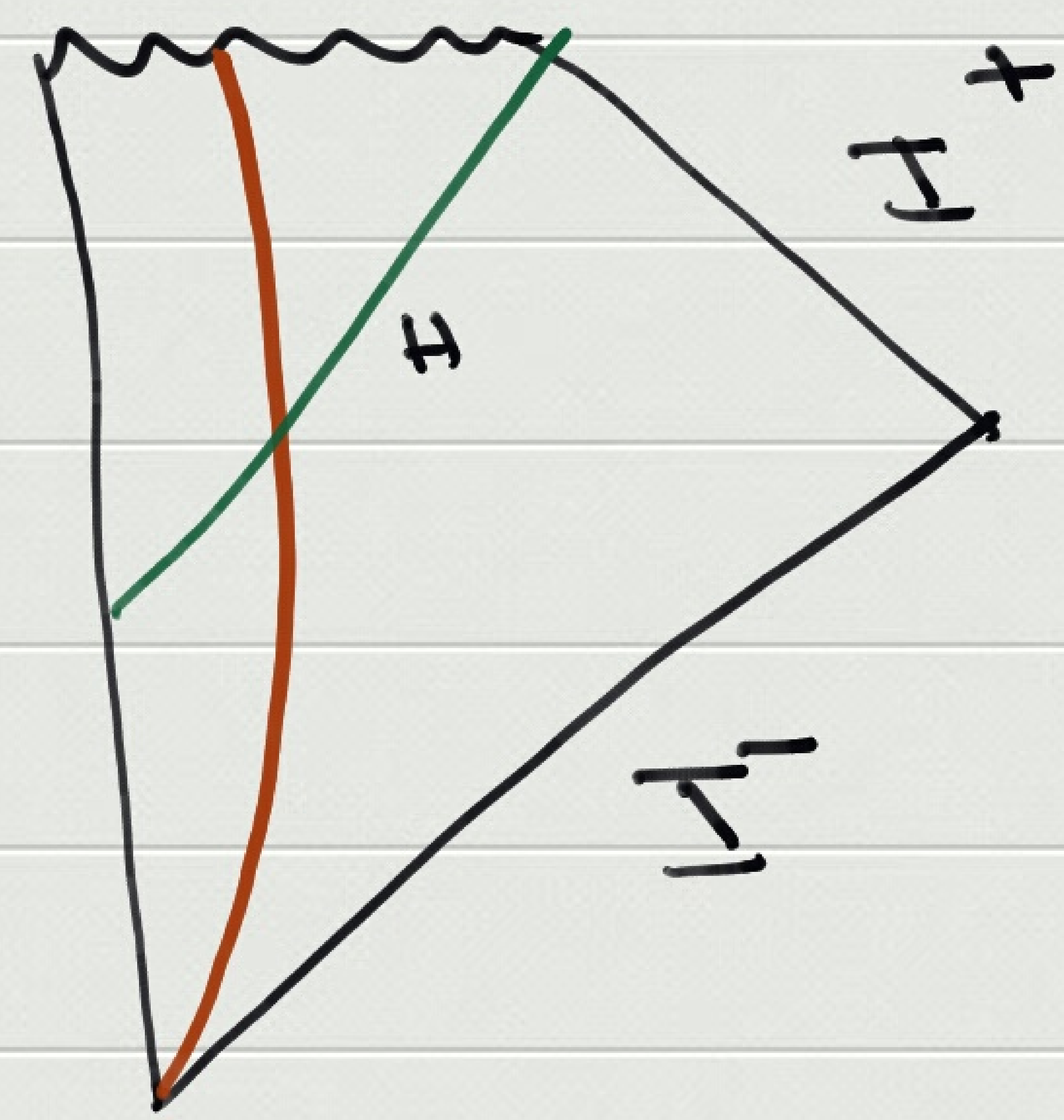
However, it is very useful to consider Hawking's original derivation as well. This goes as follows. Consider the Penrose diagram of a collapsing B.H.



We can specify initial data for the fields at past null infinity $\mathcal{I}^-$ or on $H \cup \mathcal{I}^+$

So we have two possible expansions

$$\phi = \sum a_i f_i (r, t, \Omega) + h.c.$$

specified by demanding that at past null infinity, this becomes

$$\phi = \sum_m \int \frac{a_w}{\sqrt{w}} e^{-iwv} Y_m(\Omega) + h.c.$$

The second expansion is

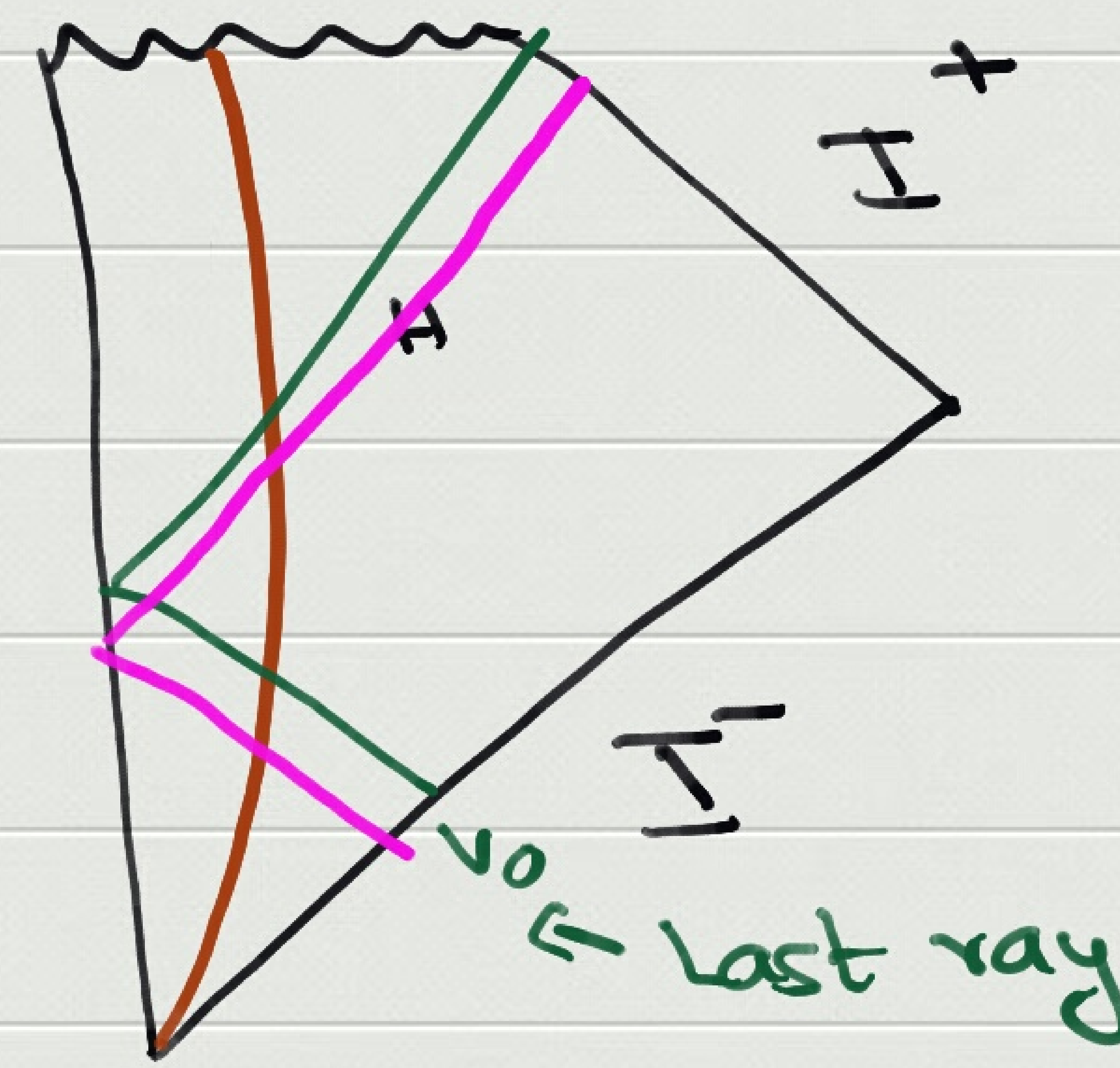
$$\phi = \sum b_i g_i (r, t, \Omega) + h.c. + c_i h_i (r, t, \Omega) + h.c.$$

where the g_i have support at future null infinity so that there

$$\phi \to \sum_m \int \frac{b_\omega}{\sqrt{\omega}} e^{-i\omega U} Y_m(\Omega) + h.c.$$

Both the c modes and the m index does not have much of a role to play in what follows.

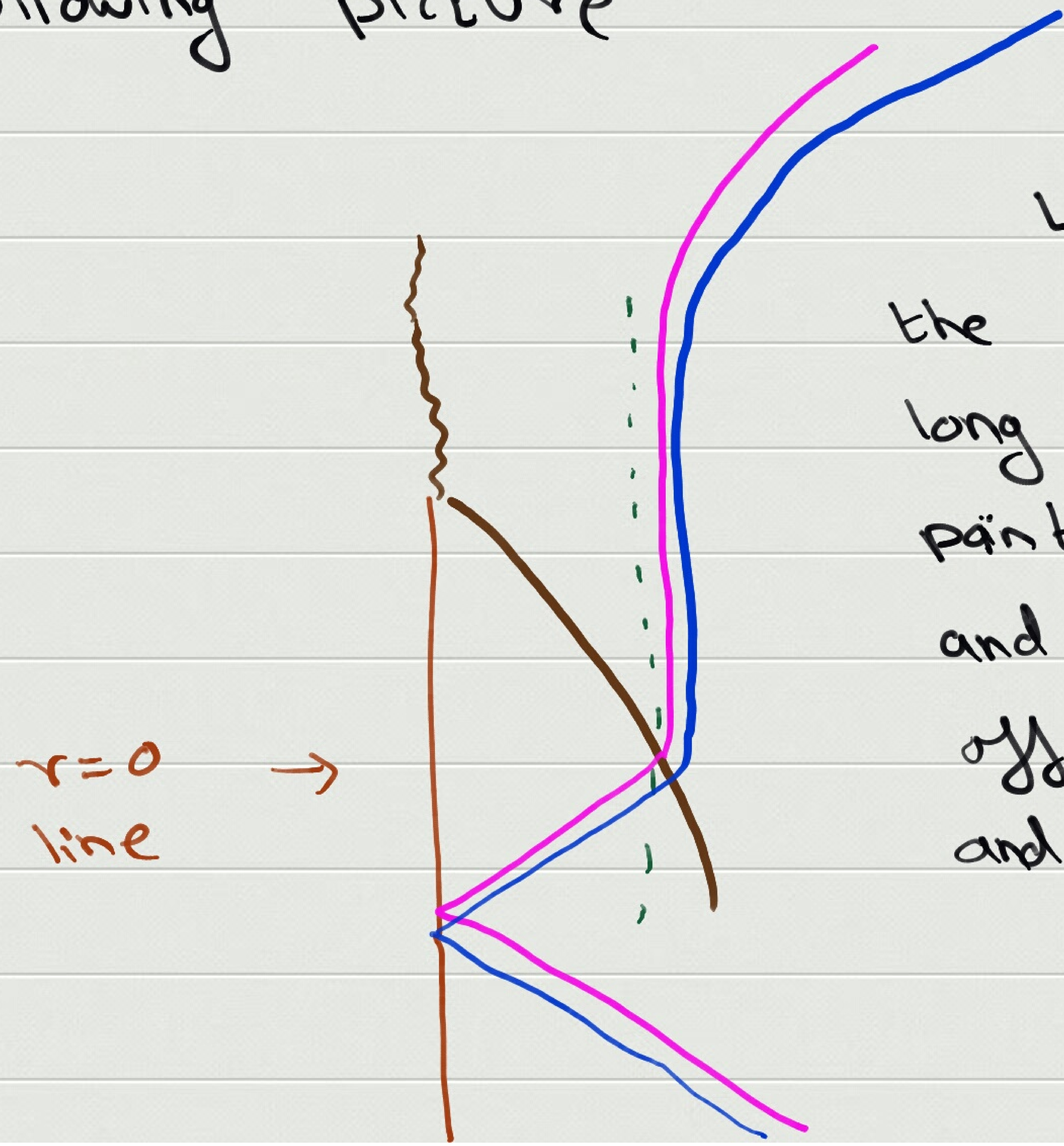
Consider the Penrose diagram again



We see modes on $\mathcal{I}^+$ can be related to modes with support just befor the last ray at v_0 using

RAY TRACING

To make this more physical, we can view this in the following picture



Light rays get stuck at the event horizon for a long time. Then at some point they enter the shell and propagate quickly, bounce off the origin of polar coordinates and back out to infinity.

Outside the shell, the trajectory of the light ray is along

$$\delta r_* = \delta t$$

The key point is that after it enters the shell, we have

$$\delta r = \delta \tau$$ to a good approximation.

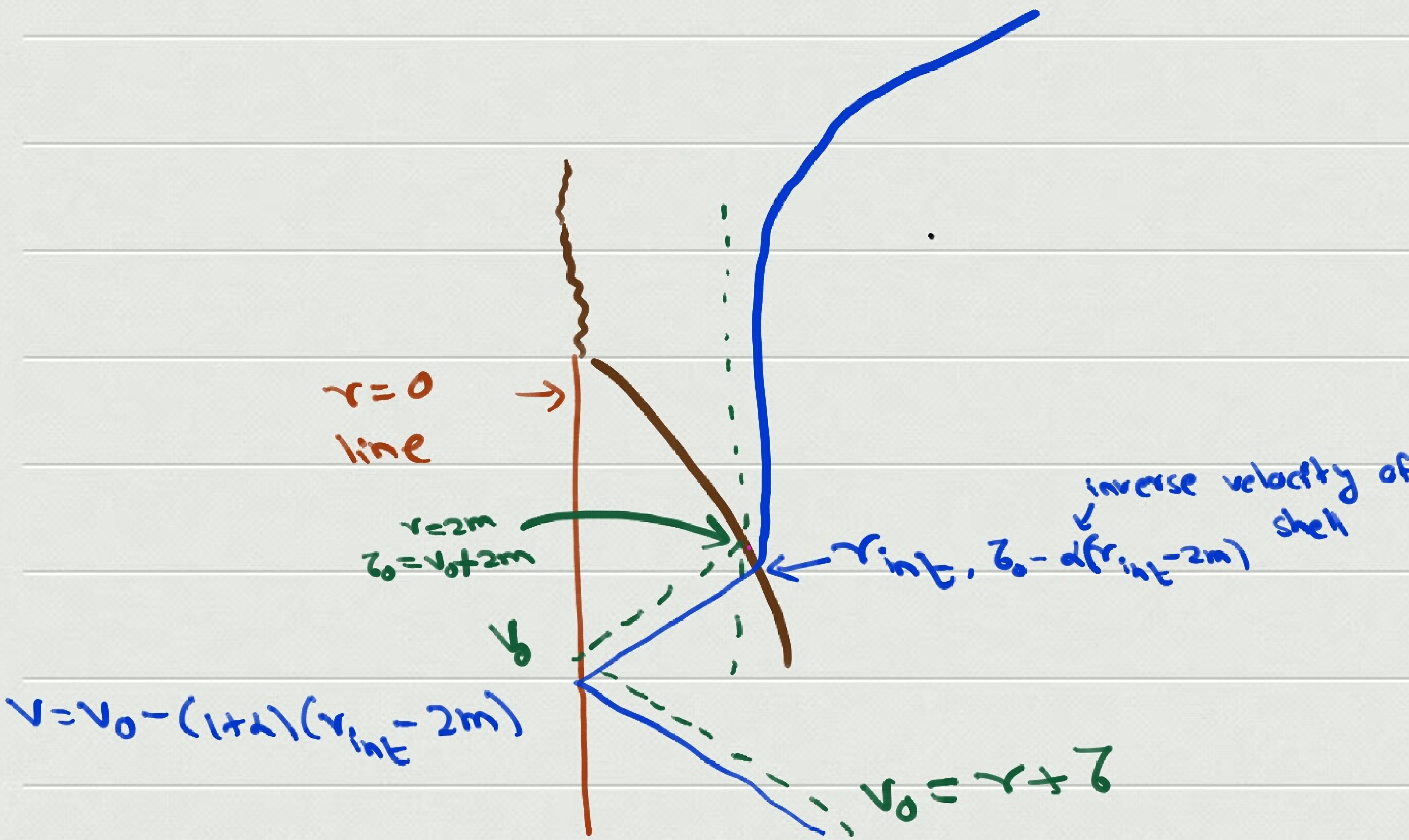
defined using radius of S^2

[this is corrected slightly, but the corrections are not important.]

When it exits the shell in the past, then the shell is larger than its event horizon and once again we use

$$|\delta r_*| = |\delta t|.$$

Now consider two rays. One is the "last ray" that gets stuck at the event horizon. The second is a ray slightly outside



From geometric considerations if v_0 is the past null coordinate of the last ray, then

$$v = v_0 - (1+\alpha)(r_{int} - 2m)$$

where r_{int} is the radius of the sphere where the ray intersects the shell.

This analysis is simple because it is done in infalling coordinates where everything is finite.

On the other hand after it exits the shell the ray stays trapped near the horizon for a long time.

There are two important effects here.

The first is that outside the shell, the ray travels along constant

$$U = t - r_* = U_{ray}$$

Second the t-coordinate of the intersection of the shell and the ray is also very large.

To see this, we see that in outside coordinates, the point of intersection is given by the intersection of

$$U = U_{ray}$$

and $V = V_S$ $\Leftarrow$ approximate trajectory of the shell near the horizon

So

$$t - r_* = U_{ray}$$
$$t + r_* = V_S$$

$$r_* = (V_S - U_{ray})/2 \Rightarrow U_{ray} = V_S - 2r_*$$

But $r_* \simeq 2m + 2m \ln \left| \frac{r_{int} - 2m}{2m} \right|$

$$\simeq 2m + 2m \ln \frac{V_0 - V}{2m(1+\alpha)}$$

Central ray tracing result.

So $U_{ray} = V_S - 4m + 4m \ln \frac{V_0 - V}{2m(1+\alpha)}$.

Aside - ray tracing in a concrete setting

We can set up the problem precisely in the Oppenheimer-Snyder solution.

$$ds^2 = -d\tau^2 + (1-\sqrt{6\pi\rho_0}\,\tau)^{4/3} dr^2 + r^2 (1-\sqrt{6\pi\rho_0}\,\tau)^{4/3} d\Omega^2 \quad , r < r_s$$

with $$T_{\tau\tau} = \frac{\rho_0}{(1-\sqrt{6\pi\rho_0}\,\tau)^2} \quad , r < r_s$$

$$ds^2 = -d\tau^2 + \frac{dr^2}{h(r,\tau)^{2/3}} + r^2 h(r,\tau)^{4/3} d\Omega^2$$

$$h(r,\tau) = \left(1 - \sqrt{6\pi\rho_0}\,\tau \frac{r_s^{3/2}}{r^{3/2}}\right)$$

Convenient to change to $x = r\,h(r,\tau)^{2/3}$, $r > r_s$; $x = (1-\sqrt{6\pi\rho_0}\,\tau)^{2/3} r$ $\quad r < r_s$

$$r = \frac{x}{(1-\sqrt{6\pi\rho_0}\,\tau)^{2/3}} \quad \text{or} \quad \left(\frac{x}{r}\right)^{3/2} = 1 - \sqrt{6\pi\rho_0}\,\tau \frac{r_s^{3/2}}{r^{3/2}} \quad \text{or } r = \left(x^{3/2} + \sqrt{6\pi\rho_0}\,\tau\, r_s^{3/2}\right)^{2/3}$$

We can now write the metric as

$$ds^2 = -g_{\tau\tau}\, d\tau^2 + 2 g_{x\tau}\, dx\, d\tau + g_{xx}\, dx^2 + x^2 d\Omega^2$$

For light rays not moving on the sphere we have

$$\left(\frac{dx}{d\tau}\right) = \frac{-g_{x\tau} \pm \sqrt{(g_{x\tau})^2 + g_{\tau\tau} g_{xx}}}{g_{xx}}$$

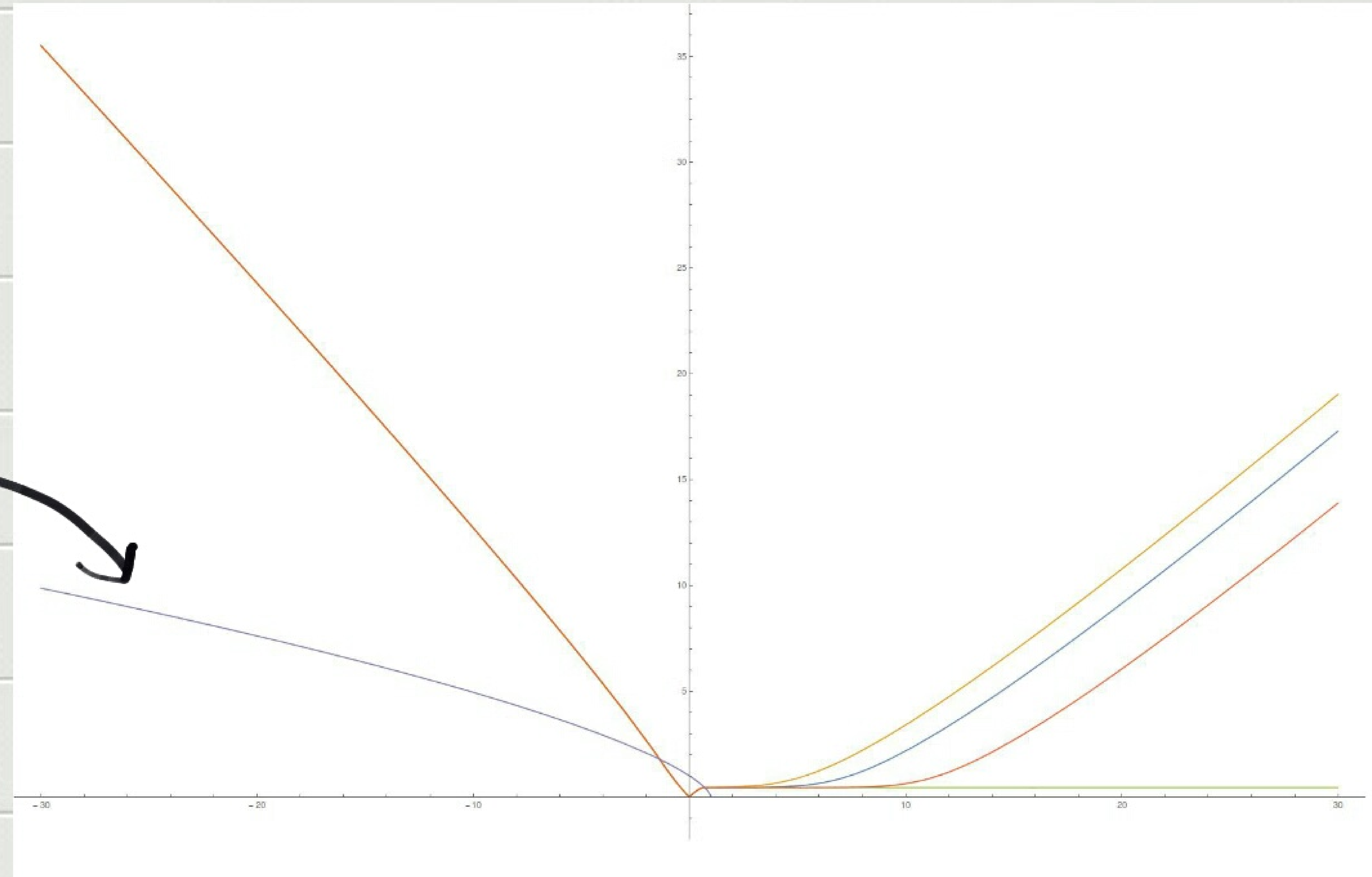
The tricky point is that we want the sign of the square root to flip once we reach $x=0$, so the ray becomes outgoing.

This can be done by cleverly arranging for negative x to represent rays that have bounced off the origin.

Now we can ask the following question. Start a particle at negative x and let it come to positive x. Calculate time elapsed for an observer at const. x.

See Mathematica file oppenheimer_geodesic.nb for the solution to this problem.

surface of star



$\uparrow x =$ radius of sphere

$\to \tau$

For some choice of parameters the geodesics look as above

So we have again ended up with an exponential map between the coordinates.
We have the outgoing modes

$$\phi_{out} = \int \frac{d\omega}{\sqrt{\omega}} \left[\frac{e^{-i\omega U}}{\Omega} a_\omega + h.c \right]$$

and the modes at past null infinity

$$\phi_{in} = \int \frac{d\omega}{\sqrt{\omega}} \left[\frac{e^{-i\omega V}}{\Omega} b_\omega + h.c. \right]$$

and once again

$$U \sim 4M \ln|v - v_0|$$ ← Note the additive constants get absorbed in phases of modes and don't matter.

So once again we see that the vacuum for the b-modes is like a thermal bath for the a-modes.

$$\rho_{\mathcal{I}^+} \sim \frac{1}{Z} e^{-\beta H}$$

with $\beta = 8\pi M$.

Now we turn to a much more precise derivation of Hawking radiation.

Conside the two-point correlation of fields across the horizon.

$$\langle \phi(U_1, V_1)\, \phi(U_2, V_2) \rangle$$

where we may have $U_1, U_2 < 0$ or $U_2 < 0$ [outside] and $U_1 > 0$.

As these points approach the light cone, we know

$$\langle \phi(x_1)\, \phi(x_2) \rangle = \frac{1}{g_{\mu\nu}(x_1 - x_2)^\mu (x_1 - x_2)^\nu}$$

for $|x_1 - x_2|^2 \rightarrow 0$

In a coordinate system where the metric is

$$ds^2 = k\, dU\, dV - dx^2$$

we have

$$\langle \phi(U_1, V_1, x_1)\, \phi(U_2, V_2, x_2) \rangle = \frac{1}{k\, \delta U\, \delta V - (\delta x)^2}$$

Now consider

$$\langle \partial_{U_1} \phi(U_1, V_1, x_1)\, \partial_{U_2} \phi(U_2, V_2, x_2) \rangle = \frac{-2k^2 \delta V^2}{(k\, \delta U\, \delta V - (\delta x)^2)^3}$$

Now

$$\lim_{\delta V \to 0} \langle \partial_{U_1} \phi_1\, \partial_{U_2} \phi_2 \rangle \propto \frac{\delta^2(\delta x)}{(\delta V)^2}$$

Prove in Tutorial

Proof:

Consider $$-2k^2 \int \frac{f(\delta\vec{x})\ \delta v^2}{(k\,\delta u\,\delta v - (\delta\vec{x})^2)^3}\, d^2\delta x$$

define $\delta\vec{x} = \vec{t}\sqrt{\delta u\,\delta v}$

$$d^2\delta x = d^2\vec{t}\,(\delta u\,\delta v)$$

$$\frac{1}{(k\,\delta u\,\delta v - (\delta x)^2)^3} = \frac{1}{(\delta u\,\delta v)^3\,(k - t^2)}$$

So the integral becomes

$$-2k^2 \int \frac{f(\delta u \delta v t)\,(\delta v)^3\,\delta u}{(\delta u\,\delta v)^3}\,\frac{d^2 t}{(k-t^2)} \xrightarrow{\delta v \to 0} \frac{-2k^2 f(0)}{(\delta v)^2}\int \frac{d^2 t}{(k-t^2)}$$

The transverse delta function shows ultra-locality in the transverse directions.

Now we try to reproduce this using the mode expansion and see what it implies.
When we solve the wave-equation, we find

$$\phi = \int \left(\frac{a_\omega}{\sqrt{\omega}} V^{-i\frac{\beta\omega}{2\pi}} + \frac{b_\omega}{\sqrt{\omega}} (-U)^{-i\beta\omega/2\pi} + h.c. \right) e^{ik\cdot x}, \quad \text{outside}$$

Note sign; this is exactly like the region II expansion.

assumes past horizon is also smooth.

otherwise form may be different.

$$\phi = \int \left(\frac{a_\omega}{\sqrt{\omega}} V^{-i\frac{\beta\omega}{2\pi}} + \frac{\tilde{b}_\omega}{\sqrt{\omega}} U^{i\beta\omega/2\pi} + h.c. \right) e^{ik\cdot x}, \quad \text{inside}$$

Now taking $v_2 < 0$, $v_1 > 0$ we see

from derivative

$$\lim_{\delta v \to 0} \langle \partial_{v_1} \Phi(v_1, v_1, x_1)\, \partial_{v_2} \Phi(v_2, v_2, x_2) \rangle \propto \delta(x_1 - x_2)\, \beta^2 / (4\pi^2 v_1 v_2)$$

$$\int d\omega\, d\omega' \sqrt{\omega\omega'} \langle \left(b_\omega (-v_2)^{-i\beta\omega/2\pi} + b^+_\omega (-v_2)^{i\beta\omega/2\pi} \right) \left(\tilde{b}_{\omega'} v_1^{i\beta\omega'/2\pi} + \tilde{b}^+_{\omega'} v_1^{-i\beta\omega'/2\pi} \right) \rangle$$

Now, we claim

$$\langle b_\omega \tilde{b}_{\omega'} \rangle = \frac{e^{-\beta\omega/2}}{1 - e^{-\beta\omega}} \delta(\omega - \omega')$$

does the job.

Check: we get for the ω-integral

$$\int d\omega \; \underbrace{\omega \, \frac{e^{-\beta\omega/2}}{1-e^{-\beta\omega}}}_{\text{even under } \omega \to -\omega} \left[\left(\frac{U_1}{-U_2}\right)^{i\beta\omega/2\pi} + \left(\frac{-U_2}{U_1}\right)^{i\beta\omega/2\pi} \right]$$

$$= \int_{-\infty}^{\infty} d\omega \; \omega \, \frac{e^{-\beta\omega/2}}{1-e^{-\beta\omega}} \left(\frac{U_1}{-U_2}\right)^{i\beta\omega/2\pi}$$

If $|U_2| < |U_1|$, we complete the contour through the upper half-plane and otherwise through the lower half-plane picking up the poles at $\beta w = 2n\pi i$ we find

$$\int w \frac{e^{-\beta w/2}}{1-e^{-\beta w}} = -\frac{1}{\beta} \sum_n n \left(\frac{v_1}{-v_2}\right)^n (-1)^n$$

from sign of contour

$e^{-\beta w/2}$ at $\beta w = 2n\pi i$

$$= \frac{-1}{\beta} \frac{v_1 v_2}{(v_1 - v_2)^2}$$

$\left[\text{Using } \sum_1^\infty n x^n = \frac{x}{(1-x)^2} \right]$

Restoring the other factors we find

$$\lim_{\delta r \to 0} \langle \partial_1 \phi_1 \, \partial_2 \phi_2 \rangle \propto \delta(x_1 - x_2) / (v_1 - v_2)^2$$

exactly as desired.

Similarly, we find [Tutorial]

$$\langle b_\omega b_{\omega'}^\dagger \rangle = \frac{1}{1-e^{-\beta\omega}} \delta(\omega-\omega').$$

$$\langle b_\omega^\dagger b_{\omega'} \rangle = \frac{e^{-\beta\omega}}{1-e^{-\beta\omega}} \delta(\omega-\omega')$$

This derivation is precise, and free of the difficulties in earlier ones.

We see the precise assumption is that the future horizon is smooth and the precise output is that low-point correlators are thermal!

References

This material is usually not covered in textbooks. See Hawking, "particle creation by black holes", 1975 for ray-tracing.

See K.P. & S.R., arXiv: 1503.08825, Section 4.1 for the correlator derivation.