31 March 2021

Lecture 21: Paradoxes for large AdS black holes

We now consider large black holes in global AdS.

For such black holes

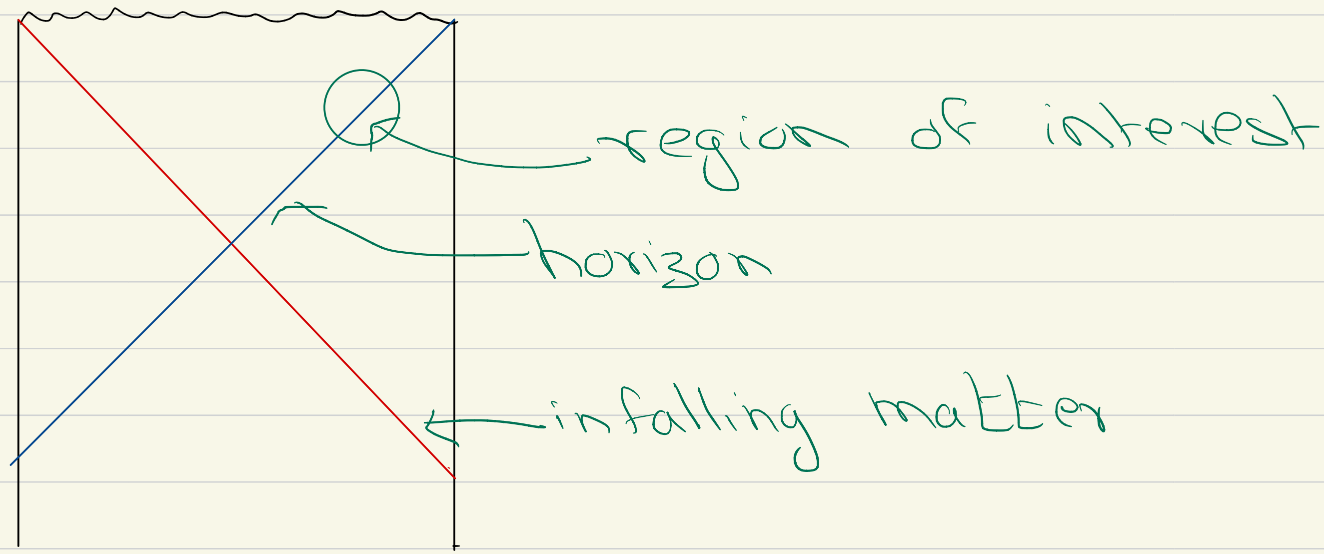
$$ds^2 \xrightarrow[t \to \infty]{} -f(r)\,dt^2 + dr^2 f^{-1}(r) + r^2 d\Omega_{d-1}^2$$

where

$$f(r) = 1 - \frac{\mu}{r^{d-2}} + r^2$$

$$\left[\mu = 8\pi^{\frac{2-d}{2}} \, \Gamma\left(\frac{d}{2}\right) \frac{G M}{d-1} \right]$$

The picture we have in mind is as follows

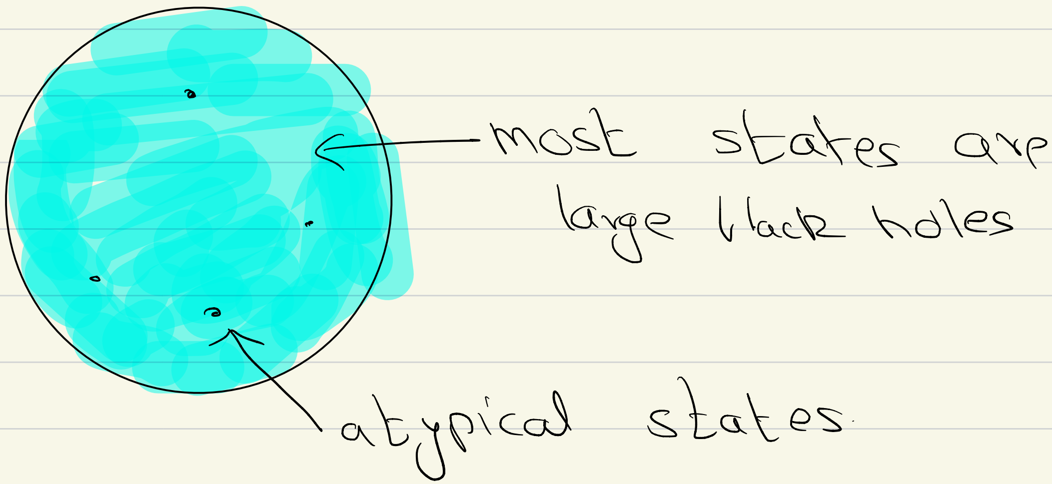


We will later consider eternal black holes but, for now, we are thinking of pure states.

Difference between large AdS black holes and evaporating black holes

Large AdS black holes dominate the microcanonical ensemble.

If one takes a **typical state** at high energies, in a theory in AdS, it corresponds to a large b.h.



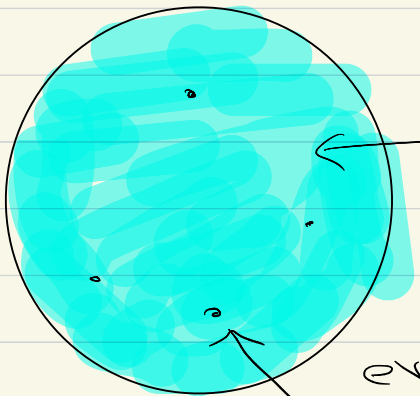
evaporating b.h. are just the opposite.
(s')

the entropy of Hawking radiation > entropy
of b.h. (s)

so if pick a state "at random" with
prob

$$1 - e^{S - S'}$$

it will be a gas of Hawking
radiation.



most states are
gas of gravitons/other particles

evaporating
black holes.

large b.h. has positive specific heat

evaporating b.h. has negative specific heat

This will be important below as we will take **a trace** over the microcanonical ensemble to obtain our paradoxes.

The main takeaway is the following. The following two statements are inconsistent:

1) Typical pure states at high-enough energies are black holes with empty interiors where one can use EFT

2) Dof in the interior are described by the same operators in the CFT for different microstates.

↑

[Explain. May seem like an innocuous assumption but will turn out to be important.]

Let us recall what effective field theory tells us

Consider a bulk scalar field propagating in the b.h. geometry

By smearing the field appropriately we can extract

1) Modes O_ω by smearing the field outside the horizon

2) Modes $\tilde{O}_\omega$ by Smearing the field inside the horizon

We must have

$$\langle\psi|\tilde{a}_\omega \tilde{a}_\omega^\dagger|\psi\rangle = \langle\psi| a_\omega a_\omega^\dagger |\psi\rangle = \frac{1}{1-e^{-\beta\omega}}$$

$$[a_\omega, a_\omega^\dagger] = 1$$

$$[\tilde{a}_\omega, \tilde{a}_\omega^\dagger] = 1$$

$$\langle\psi| a_\omega \tilde{a}_\omega |\psi\rangle = \frac{e^{-\beta\omega/2}}{1-e^{-\beta\omega}}$$

$$[a_\omega, \tilde{a}_\omega] = 0$$

[Emphasize that this follows from a 2-pt Fn across the horizon.]

Furthermore

$$[H, a_\omega] = -\omega a_\omega$$

$$[H, \tilde{a}_\omega] = \omega a_\omega$$

This is important so let us briefly understand how this is derived.

Negative occupancy paradox

We stated that in typical states

$$\langle \Psi | \tilde{a}_\omega \tilde{a}_\omega^\dagger | \Psi \rangle = \frac{1}{1 - e^{-\beta\omega}}$$

and

$$[H, \tilde{a}_\omega] = \omega \tilde{a}_\omega$$

We also know [purely based on kinematics] that expectation values in a typical state are close to thermal expectation values.

[This also uses equivalence of ensembles, and we are not writing $\frac{1}{\sqrt{S}}$ terms here.]

So, <u>we need</u>

$$T = \frac{1}{Z(\beta)} \operatorname{tr}\left(e^{-\beta H} \tilde{a}_\omega \tilde{a}_\omega^{+}\right) = \frac{1}{1-e^{-\beta\omega}}$$

Now, using the cyclicity of the trace

$$T = \frac{1}{Z(\beta)} \operatorname{tr}\left(e^{-\beta H} \tilde{a}_\omega \tilde{a}_\omega^{+}\right) = \frac{1}{Z(\beta)} \operatorname{tr}\left(\tilde{a}_\omega^{+} e^{-\beta H} \tilde{a}_\omega\right)$$

Using the $H, \tilde{a}_\omega^{+}$ commutator

$$T = \frac{1}{Z(\beta)} \operatorname{tr}\left(e^{-\beta H} \tilde{a}_\omega^{+} \tilde{a}_\omega\right) e^{-\beta\omega}$$

$$= \frac{1}{Z(\beta)} e^{-\beta\omega} \operatorname{tr}\left(e^{-\beta H} \left(\tilde{a}_\omega \tilde{a}_\omega^{+} - 1\right)\right)$$

so

$$T = e^{-\beta w} T - 1$$

so

$$T = -\frac{e^{-\beta w}}{1 - e^{-\beta w}}$$

But T is the expectation value of a positive operator!

so this negative result is absurd!

What does the paradox imply?

1) It suggests that if we look for <u>fixed</u> $\tilde{\mathcal{O}}_\omega$ that behave correctly in typical states, we run into a paradox.
[Does <u>not</u> <u>rule</u> out $\tilde{\mathcal{O}}_\omega$ that behave correctly in a small fraction of states]
2) It was used to argue for

a) either large black holes in AdS have a firewall behind their horizon.

b) The boundary theory does not describe the interior of typical states and must be supplemented with additional dof

3) Note that holography of information/complementarity is **insufficient** to resolve this.

We made no assumption about the factorization of the Hilbert space.

4) This paradox **is inapplicable** (as it stands) to black holes in flat space or to small black holes in AdS.

These never dominate the ensemble]

[Emphasize again, by reviewing the trace argument]

So it is possible this paradox is a problem in AdS and not for more realistic black holes.

The infalling number operator

Consider the operator

$$N_a = \frac{1}{1-e^{-\beta\omega}}\Big[\left(a_\omega^{\dagger} - e^{-\beta\omega/2}\tilde{a}_\omega\right)\left(a_\omega - e^{-\beta\omega/2}\tilde{a}_\omega^{\dagger}\right) + \left(\tilde{a}_\omega^{\dagger} - e^{-\beta\omega/2}a_\omega\right)\left(\tilde{a}_\omega - e^{-\beta\omega/2}a_\omega^{\dagger}\right)\Big]$$

Using

$$\left(\tilde{a}_\omega - e^{-\beta\omega/2}a_\omega^{\dagger}\right)|\psi\rangle = 0 = \left(a_\omega - e^{-\beta\omega/2}\tilde{a}_\omega^{\dagger}\right)|\psi\rangle$$

for states with a smooth horizon, we expect

$$\langle\psi|N_a|\psi\rangle = 0$$

For states with a smooth horizon.

Now consider eigenstates of the Schwarzschild number operator

$$N_\omega = a^\dagger_\omega a_\omega$$

$$N_\omega |n\rangle = n|n\rangle$$

Notice that

$$\langle n| a_\omega \tilde{a}_\omega |n\rangle = 0$$

This is because

$$N_\omega \tilde{a}_\omega |n\rangle = n \tilde{a}_\omega |n\rangle \quad \left[\begin{matrix} [N_\omega, \tilde{a}_\omega] \\ = 0 \end{matrix} \right]$$

but

$$N_\omega a^\dagger_\omega |n\rangle = (n+1) a^\dagger_\omega |n\rangle$$

Also $\langle n | \tilde{a}_\omega \tilde{a}_\omega^+ | n \rangle \geq 1$

$\langle n | a_\omega a_\omega^+ | n \rangle \geq 1$

$$\langle n | N_a | n \rangle = \langle n | \frac{1}{1-e^{-\beta\omega}} \Big[(a_\omega^+ - e^{-\beta\omega/2} \tilde{a}_\omega)(a_\omega - e^{-\beta\omega/2} \tilde{a}_\omega^+) + (\tilde{a}_\omega^+ - e^{-\beta\omega/2} a_\omega)(\tilde{a}_\omega - e^{-\beta\omega/2} a_\omega^+) \Big] | n \rangle$$

$$\geq \frac{1}{1-e^{-\beta\omega}} \, 2 e^{-\beta\omega}$$

But now note that

$[H, N_\omega] = 0$ [neglecting small corrections]

Now consider a trace over the micro-ens.

$$\langle \psi | N_a | \psi \rangle = \frac{1}{e^S} \sum_{E \in E_0 - \Delta}^{E \in E_0 + \Delta} \langle E | N_a | E \rangle = 0$$

But since $[N_\omega, H] = 0$, we can perform a <u>change of basis</u> in the trace and evaluate the trace in the number eigenbasis.

But

$$\frac{1}{e^S} \sum \langle n | N_a | n \rangle \geq \frac{2 e^{-\beta \omega}}{1 - e^{-\beta \omega}}$$

So we again have a contradiction.

The same comments as previously apply here.

This paradox involves a trace. It does not preclude the possibility that a small fraction of energy eigenstates have $\langle N_a \rangle = 0$.

We will later see how these paradoxes can be resolved via state-dependence.

But, we now want to emphasize an important point.

It is sometimes believed that these paradoxes afflict only single-sided black holes.

We now explore how the same paradoxes also apply to the eternal black hole.

Review of the Eternal black hole

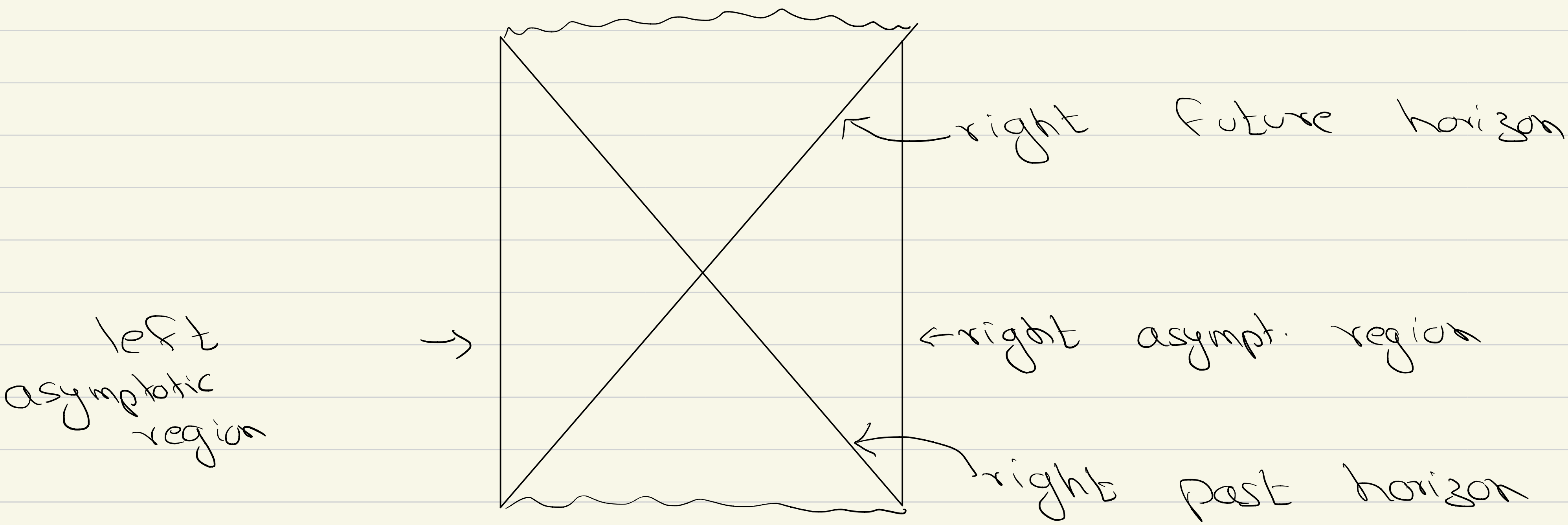
We consider the maximal extension of the geometry

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-1}^2$$

$$f(r) = 1 - \frac{\mu}{r^{d-2}} + r^2$$

Note there is no "$t \to \infty$" in the metric above.

The geometry looks like this



It is believed that this is dual to the thermofield doubled state of two CFTs

$$|\psi\rangle_{tfd} = \frac{1}{Z(\beta)^{1/2}} \sum_E e^{-\beta E/2} \, |E, E\rangle$$

where the sum runs over all eigenstates
[typo in current version of the review.
sum is missing!]

This is the canonical example of

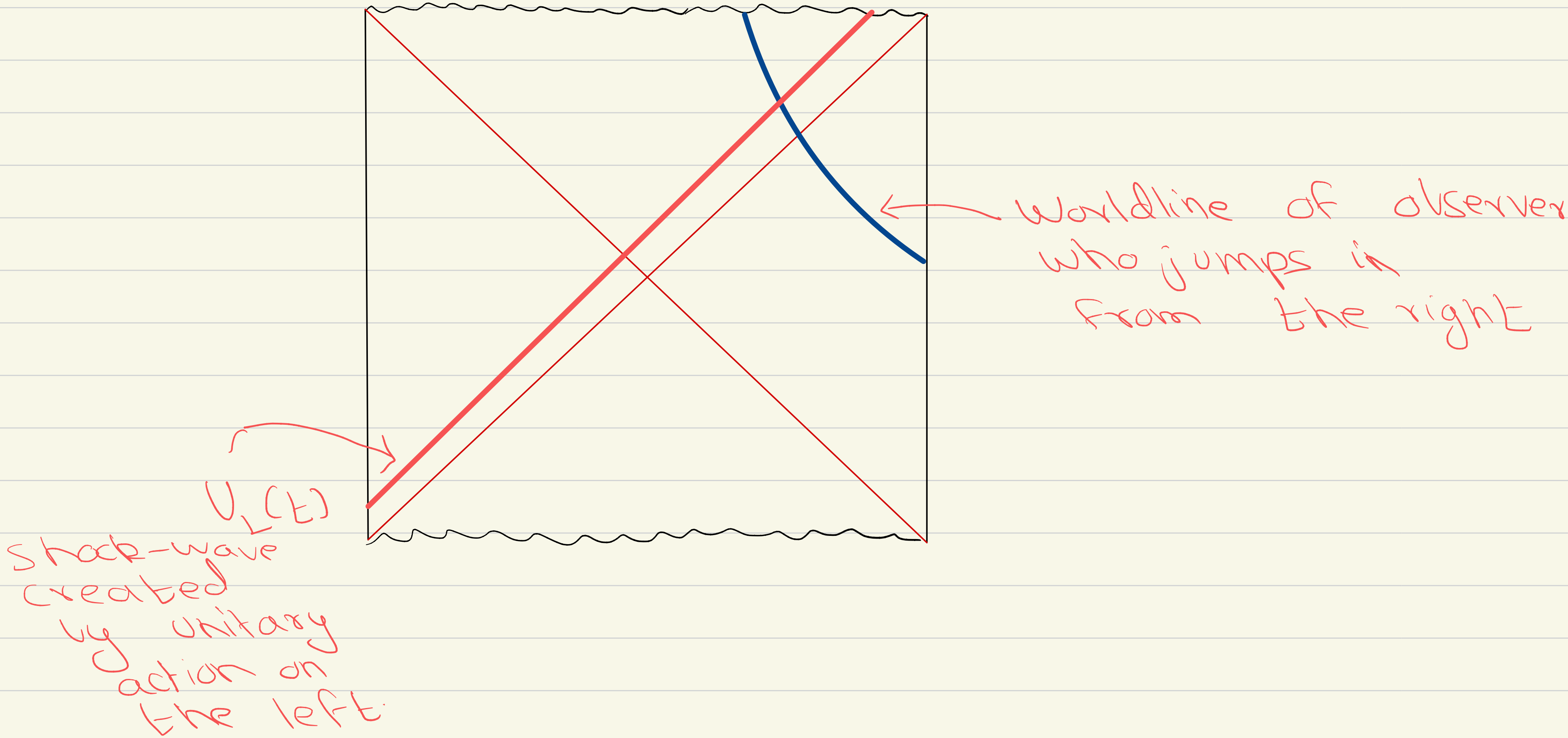
$$ER = EPR$$

Note a remarkable feature of this proposal

The two CFTs are entangled but noninteracting.

In general, when this happens, no unitary on one system can affect anything on the other system.

But here we find something interesting



The experience of a right infalling observer can be affected by a shockwave thrown from the left!

Note that this only happens in the interior. We cannot send any signals from the left boundary to the right one.

We now point out that the following 3 assumptions are inconsistent

1) Eternal black hole is dual to the thermofield double state

2) Dof in the interior are described by the same operators in the eternal b.h. and a class of states related to the eternal b.h. by Hamiltonian time evolution

3) Disentangled states $|E, E\rangle$ are not connected by a wormhole.

↑

Explain last point.