Quantum Aspects of Black Hole: Lecture 11; Kerr black holes.

Lecture 11

We now move on to a discussion of the Kerr solution, which is the most general stationary black hole solution and has both charge and angular momentum.

$$ds^2 = -\frac{(\Delta - a^2\sin^2\theta)}{\Sigma}dt^2 - \frac{2a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma}dt\,d\phi$$

$$+ \left[\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right]\sin^2\theta\, d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma\, d\theta^2$$

$$A_\mu = \frac{-er}{\Sigma}\left[(dt)_\mu - a\sin^2\theta(d\phi)_\mu\right]$$

$$\Sigma = r^2 + a^2\cos^2\theta$$

$$\Delta = r^2 + a^2 + e^2 - 2Mr$$

Kerr black holes are astrophysically significant because the dynamics of accretion disks tends to add angular momentum to the black hole. On the other hand charge tends to get neutralized.

It also has many interesting theoretical properties.

First note that while $\frac{\partial}{\partial t}$ is a Killing vector it is not orthogonal to surfaces of constant r, θ, ϕ. This is the statement that the Kerr solution is stationary but not static.

Verify that this is a solution using Mathematica.
Mention of GRS-1915.

'The mass and angular momentum of this soln are M and $J = Ma$

Both of these are defined asymptotically.
and include the contribution of the gravitational field.

Our next job is to figure out the location of the horizon.

At first, one might get confused by the surface, where $g_{00} = 0$

$$g_{00} = 0 \Rightarrow \Delta - a^2 \sin^2\theta = 0$$

But this is not a horizon since light rays can move in and out of this region.

To do this, it is useful to rewrite the metric as

$$ds^2 = \frac{-\Delta}{\Sigma}\left(dt - a\sin^2\theta\, d\phi\right)^2 + \frac{\sin^2\theta}{\Sigma}\left[(r^2+a^2)d\phi - a\,dt\right]^2$$
$$+ \frac{\Sigma}{\Delta}dr^2 + \Sigma\, d\theta^2$$

Now in this region, unless $\theta = 0$, we have $\Delta > 0$.
Now consider a trajectory so that

$$\frac{d\phi}{d\lambda} = \frac{a}{r^2+a^2}\frac{dt}{d\lambda}.$$

Now we see that we can have null trajectories with $\frac{dr}{d\lambda} = 0$ or $\frac{dr}{d\lambda} < 0$ or $\frac{dr}{d\lambda} > 0$

In particular a light ray can enter this region and also exit it.

The boundary $\Delta - a^2 \sin^2\theta = 0$ marks a region called the ergosphere; but not the horizon.

The horizon is at

$$\Delta = 0$$

or

$$r^2 + a^2 + e^2 - 2Mr = 0$$

Once we cross into the region where $\Delta < 0$, we see that the only timelike coordinate is r. So a light ray which has $\frac{dr}{dt} < 0$ as it crosses $\Delta = 0$ cannot reverse itself and come out

As in the R.N. case, we see the Kerr black hole has two horizons at

$$r_\pm = M \pm \sqrt{M^2 - (a^2+e^2)}$$

There is an extremal limit when $M^2 = a^2 + e^2$ and $r_+ = r_-$

The GRS 1915 B.H. is close to being extremal!

We now set $e=0$ to lighten the notation.

Now let us consider the near-horizon geometry. At first sight, this is confusing but the limit is facilitated by taking

$$\psi = \left(\phi - \frac{a}{r_+^2 + a^2} t\right) \equiv (\phi - \Omega t)$$

$$\Omega = \frac{a}{r_+^2 + a^2}$$

Now the near horizon limit becomes

$$ds^2 = -\frac{(r_+ - r_-)(r - r_+)}{r^2 + a^2\cos^2\theta}\left(1 - \frac{a^2\sin^2\theta}{r_+^2 + a^2}\right)^2 dt^2$$

$$+ \left(\frac{\sin^2\theta}{r^2 + a^2\cos^2\theta} + O[r - r_+]\right) d\psi^2$$

$$+ \frac{(r^2 + a^2\cos^2\theta)\,dr^2}{(r_+ - r_-)(r - r_+)} + (r^2 + a^2\cos^2\theta)\,d\theta^2$$

The r-t part near the horizon is

$$ds^2 = (r_+^2 + a^2\cos^2\theta)\left[-\frac{(r_+ - r_-)(r - r_+)\,dt^2}{(r_+^2+a^2)^2} + \frac{dr^2}{(r_+ - r_-)(r - r_+)}\right]$$

$$+ \frac{\sin^2\theta}{r_+^2 + a^2\cos^2\theta}\,d\psi^2 + (r_+^2 + a^2\cos^2\theta)\,d\theta^2$$

This is Rindler space fibred on the S^1 of the θ- coordinate.

Nevertheless, we can now easily transform to coordinates that make the region near $r \sim r_+$ regular.

with

$$dr_*^2 = \frac{dr^2 \, (r_+^2 + a^2)^2}{(r_+ - r_-)^2 (r - r_+)^2} \Rightarrow r_* = \frac{(r_+^2 + a^2)}{(r_+ - r_-)} \ln\left|\frac{r - r_+}{r_+}\right|$$

We now transform to

$$U = -e^{\alpha(r_* - t)} \; ; \quad V = e^{\alpha(t + r_*)}$$

and so

$$dU\,dV = \alpha^2 e^{2\alpha r_*} (dt^2 - dr_*^2).$$

but

$$e^{2\alpha r_*} = (r - r_+)^{2\alpha(r_+ + a^2)/(r_+ - r_-)}$$

So setting $\boxed{\alpha = \dfrac{r_+ - r_-}{2(r_+^2 + a^2)}}$ makes the horizon regular.

We can also write

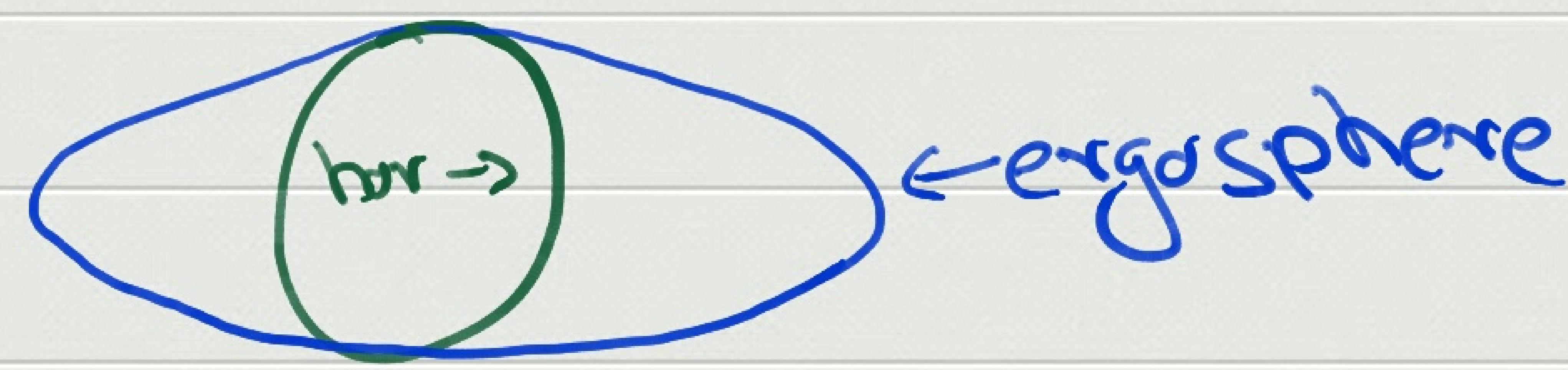
$$\alpha = \frac{(M^2 - a^2 - e^2)^{1/2}}{2M\left[M + (M^2 - a^2 - e^2)^{1/2}\right] - e^2}$$

which is the form you will see in Wald.

Recall $\dfrac{\alpha}{2\pi}$ is the temperature!

In the extremal limit, the temperature $\longrightarrow 0$.

We now return to the region where $g_{tt} < 0$ but still outside the horizon.



This is called the ergosphere.

In the ergosphere

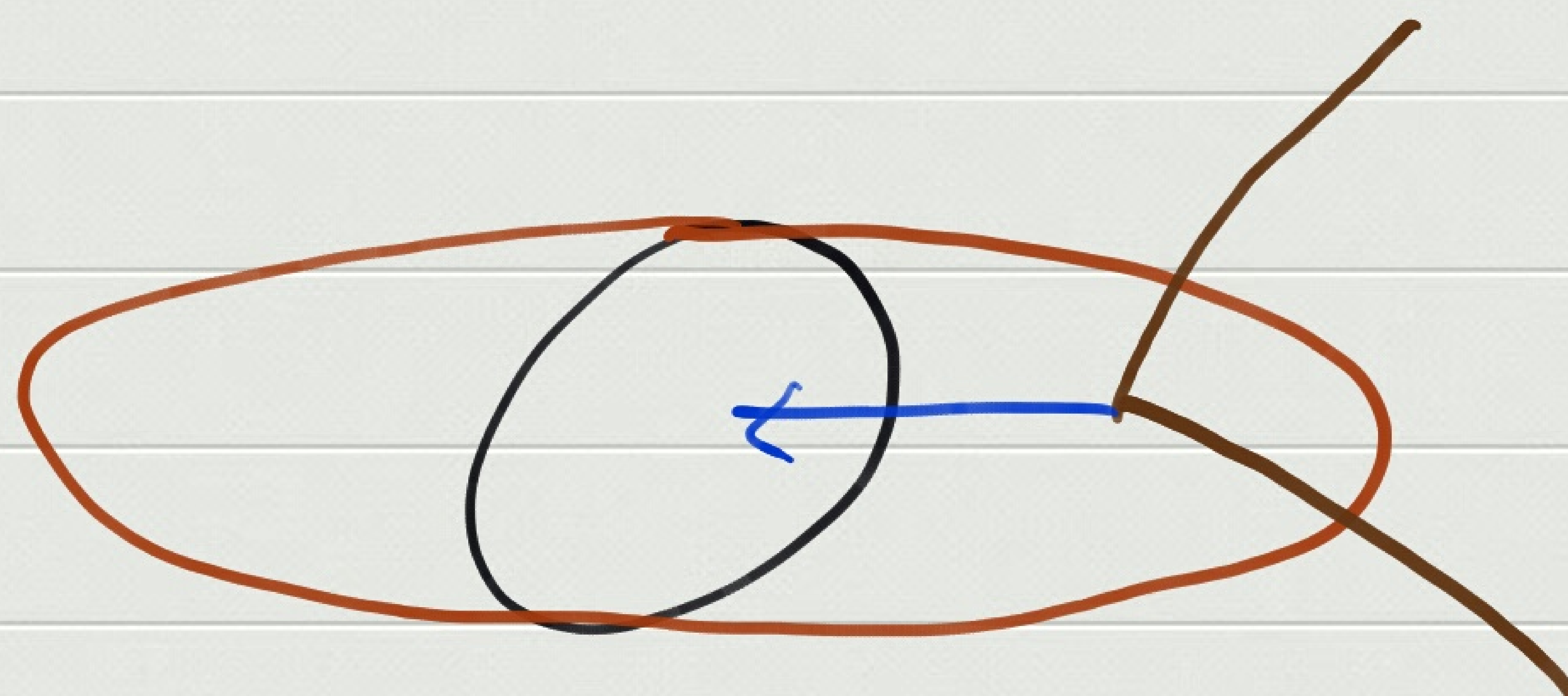
$$\frac{\partial}{\partial t}$$

becomes spacelike.
But we have another killing vector $\frac{\partial}{\partial \phi}$

The vector $\chi = \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi}$

is a timelike killing vector in the ergosphere.

The existence of the ergosphere allows the Penrose process to extract energy from the black hole!
The idea is as follows. Consider a particle that enters the ergosphere, breaks into two parts and one part exits the ergosphere



We have two Killing vectors $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi}$ and consequently the particle's momentum in these directions is conserved. So

note minus sign to make E +ve

$$-m_0 \frac{dx^\mu}{dt} K^\nu g_{\mu\nu} = E \quad \text{(energy)}$$

and $$m_0 \frac{dx^\mu}{dt} R^\nu g_{\mu\nu} = L \quad \text{(angular momentum)}$$

$$K = \frac{\partial}{\partial t} \quad ; \quad R = \frac{\partial}{\partial \phi}$$

Now when the particle breaks into two, we have

$$E^{(1)}_{in} = E^{(1)} + E^{(2)}$$
$$L^{(1)}_{in} = L^{(1)} + L^{(2)}$$

But notice that for particle 2 can have locally positive energy, we need

$$-\frac{dx_2^{\mu}}{d\tau}\left(k^{\nu} + \Omega_H R^{\nu}\right) g_{\mu\nu} > 0$$

and therefore we see that

$$E^{(2)} - \Omega_H L^{(2)} \geq 0$$

[relative minus sign because of the definitions of E, L]

But this means we can have

$$E^{(2)} < 0 \;!$$

and so $E^{(1)} > E^{(1)}_{in}$

this is called energy extraction from rotating black holes!