QABH Lecture 14

Last time we introduced the surface gravity through

$$\nabla_\mu (\chi^a \chi_a) = -2k\, \chi_\mu$$

or equivalently $\quad \chi^a \nabla_a \chi_\mu = k\, \chi_\mu .$

k measures the failure of the Killing parameter to be an affine parameter

$$\frac{d\lambda}{dv} = e^{kv} \qquad \text{or} \qquad \lambda = \frac{1}{k} e^{kv}$$

Now we will do some algebraic manipulations to get another formula for κ. Don't get scared!

Recall that we proved last time that

$$\chi_{[a} \nabla_b \chi_{c]} = 0$$

since χ is hypersurface orthogonal. This follows from $\chi = f\,dg \Rightarrow d\chi = \frac{df}{f} \wedge \chi$

$\Rightarrow \quad \chi \wedge d\chi = 0.$

but since

$$\nabla_b x_c + \nabla_c x_b = 0$$

we have

$$x_a \nabla_{[b} x_{c]} = 2 x_a \nabla_b x_c$$

so $\quad x_{[a} \nabla_b x_{c]} = 0$

$$\Rightarrow x_a \nabla_b x_c + x_c \nabla_a x_b + x_b \nabla_c x_a = 0$$

contract with $\nabla^a x^b$

$$\left(x_a \nabla_b x_c + x_c \nabla_a x_b + x_b \nabla_c x_a \right) \nabla^a x^b$$

and use

$$x_a \nabla^a x_\mu = k x_\mu \quad \text{and} \quad x^\mu \nabla^a x_\mu = -k x^a$$

we get

$$x_c \left(\nabla_a x_b \right)^2 = -2 k^2 x_c .$$

or

$$k^2 = -\frac{1}{2} \left(\nabla_a x_b \right)^2$$

We can use this to derive another formula for κ.

As an identity, we have

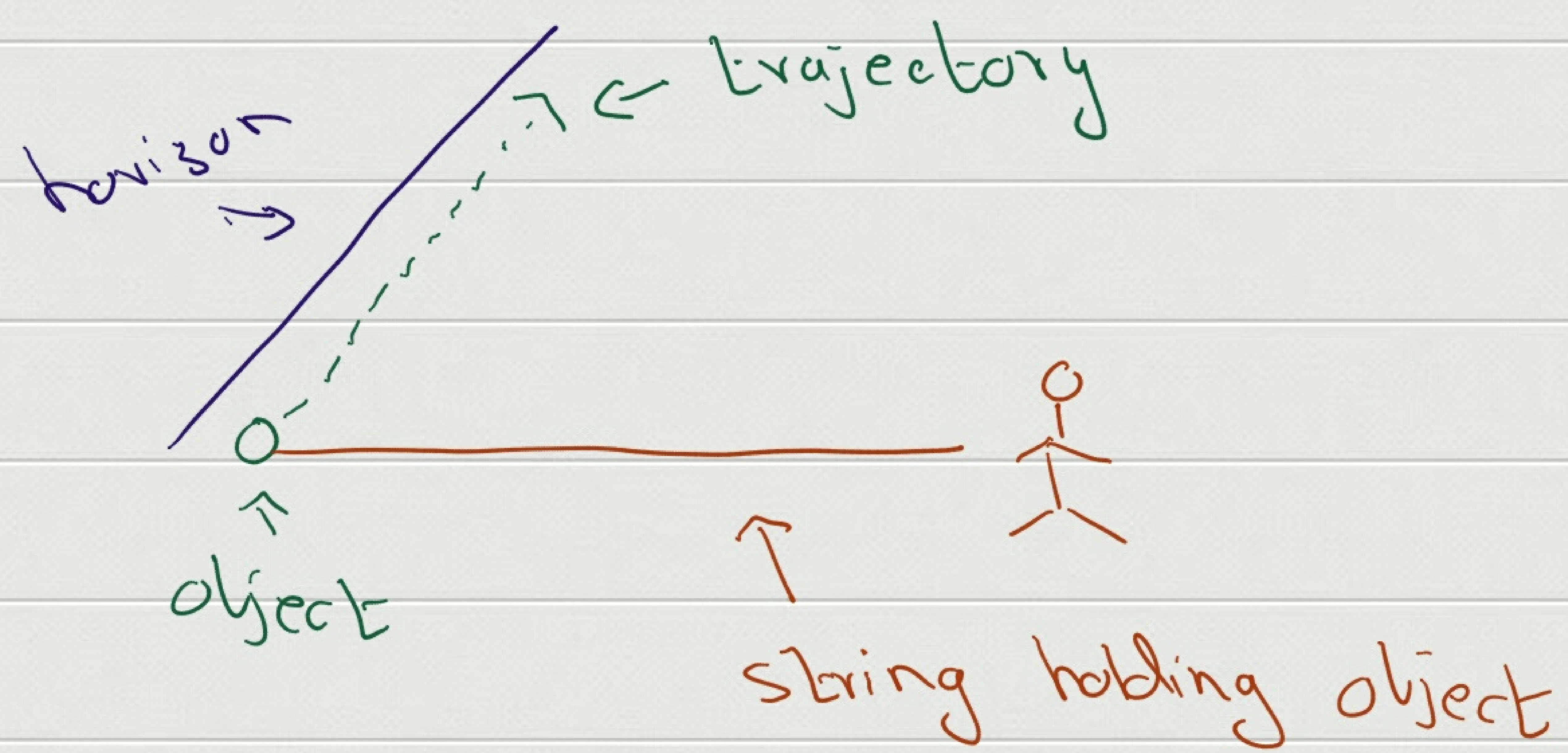
$$\chi^{[a} \nabla^b \chi^{c]} \chi_{[a} \nabla_b \chi_{c]} \propto \chi^a \chi_a (\nabla^b \chi^c)^2 - 2(\chi^a \nabla_a \chi_b)(\chi^a \nabla_a \chi^b).$$

The L.H.S. vanishes at the horizon. So we also have

$$\kappa^2 = -\lim \frac{a_b a^b}{(\chi^a \chi_a)}, \text{ with } a_b \equiv \chi^a \nabla_a \chi_b$$

this allows us to provide a simple physical interpretation for χ. Consider a STATIC BLACK HOLE.

Consider an object moving along the horizon. This is not a geodesic, so we imagine the object is held on its path by a string.



How is the 4-velocity related to the vector χ^a.
near the horizon. $\chi^a \propto U^a$ but

$$\chi^a \chi_a = -V^2$$

but $\qquad V =$ red shift factor

$$U^a U_a = -1$$

so

$$U^a = \frac{\chi^a}{V}$$

The local acceleration is $U^a \nabla_a U^b = \alpha^b$

$$= \frac{1}{V^2} \chi^a \nabla_a \chi^b$$

where I pulled V out of the derivative since $\chi^a \nabla_a V = 0$.
its magnitude is

$$|\alpha|^2 = \left| \frac{(\chi^a \nabla_a \chi_b)(\chi^c \nabla_c \chi^b)}{(\chi^a \chi_a)^2} \right|$$

what is the force the observer at infinity has to exert to maintain this acceleration.

say the observer at infinity tugs the string. The work done by him is

$$W_\infty = F_\infty \, \delta r$$

where δr is the proper distance the string moves.

The local work done on the particle is

$$W_{local} = F_{local} \, \delta r_{local}$$

but $\delta r_{local} = \delta r$ if the string is inextensible.

But we have

$$W_\infty = V \, W_{local}$$

So

$$F_\infty = V\, F_{local}$$

For a unit mass object

$$|F_\infty| = V\, |F_{local}| = V\,|\alpha^b| = \sqrt{\left|\frac{(\chi^a \nabla_a \chi_b)(\chi^c \nabla_c \chi^b)}{(\chi^\mu \chi_\mu)}\right|}$$

which is just the surface gravity.

This explains the name

We now move to the first law.
For simplicity, we consider non-rotating black-holes.

Consider throwing some matter into the B.H. The energy current is

$$j_\mu = T^\nu{}_\mu \chi_\nu$$

so the increase in mass is

$$\int j_\mu k^\mu \, d\lambda \, dA$$

where k^μ is the tangent (and the normal!)
and dA is the area element.

But previously we proved that

$$\chi^\mu = \kappa \lambda \, k^\mu$$

so

$$\Delta M = \int T_{\mu\nu} \, k^\mu \chi^\nu (\kappa\lambda) \, d\lambda \, dA$$

$$= \frac{\kappa}{8\pi G} \int R_{\mu\nu} \, k^\mu k^\nu \, \lambda \, d\lambda \, dA$$

Now for a quasi-static process, using the Raychaudhuri eqn

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma_{ab}\sigma^{ab} - R_{cd} k^c k^d$$

$$\simeq -R_{cd} k^c k^d$$

so

$$\Delta M = -\frac{\kappa}{8\pi G} \int \frac{d\theta}{d\lambda} \lambda \, d\lambda \, dA.$$

$$= \frac{\kappa}{8\pi G} \int \theta \, d\lambda \, dA.$$ ← bdry terms dropped $\theta = 0$ at beginning and end of process.

$$= \frac{\kappa}{8\pi G} \Delta A$$

Finally, we can state the 0^{th} law of thermodynamics of black holes.

For a stationary B.H., κ is constant on the horizon

$\Downarrow$

similar to temperature is constant throughout a body in thermal equilibrium

From the relation between the affine and killing parameter on the horizon

$$\lambda = \frac{1}{k} e^{kv}$$

we know k is just what entered our coordinate transformations earlier.

$$k = \frac{(M^2 - a^2 - e^2)^{1/2}}{2M\left[M + (M^2 - a^2 - e^2)^{1/2}\right] - e^2}$$

We can use this formula to discuss the analogue of the third law of thermodynamics.

One common formulation is that $S \to 0$ as $T \to 0$. This clearly does not hold.
If κ is proportional to T, we see that extremal black holes have 0 temperature but they do not have zero area.

Another formulation is that we cannot reach $T \to 0$ through a finite process and there is some evidence for this.