Lecture 3 - QFT in curved spacetime : Examples

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Last time, we discussed the general formalism of QFT in curved spacetime. Today we will discuss some specific examples.

As a first example, we will consider particle creation in a FRW cosmology. [Birrell & Davies, 3.4]

Consider a background metric

$$ds^2 = dt^2 - a^2(t)\, dx^2$$

Recall by defining $dt^2 = a^2 d\eta^2$

We can write this as

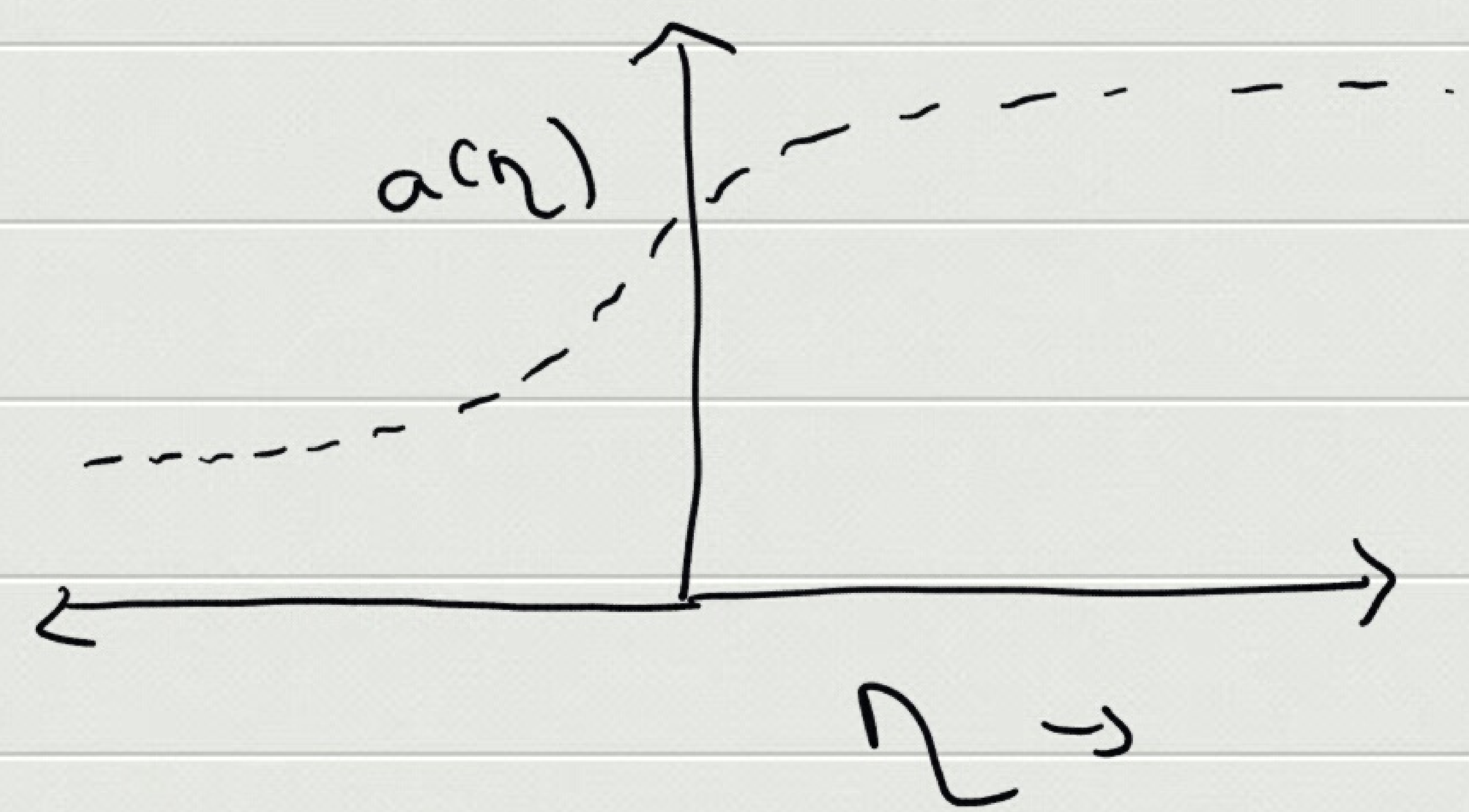
$$ds^2 = a^2(\eta)\,(d\eta^2 - dx^2).$$

We will consider a case where

$$a^2(\eta) = A + B \tanh \rho\eta$$

as $\eta \to \infty$ $\quad a(\eta) \to \sqrt{A+B} \to$ constant

$\eta \to -\infty$ $\quad a(\eta) \to \sqrt{A-B} \to$ constant



This is typical of the kinds of problems we deal with.
We are not concerned with the matter stress-energy tensor that leads to this background.
We take it as given and just consider fluctuations

It is useful to understand why particle creation takes place here. Consider an observer in the far past. She sees a field with effective

$$\text{mass}^2 \rightarrow (A-B)\,m^2$$

then the background spacetime starts changing the value of m. The modes that behave like $e^{-ik\omega}$ in the past are not the ones that behave like $e^{ik\omega}$ in the future.

Mathematically, the problem is similar to scattering off a barrier and we can solve it as follows

Now $\sqrt{-g} = a^2(\eta)$

$g^{\eta\eta} = g^{xx} = \frac{1}{a^2(\eta)}$

Recall the wave eqn is

$$\frac{1}{\sqrt{-g}} \partial_\mu g^{\mu\nu} \partial_\nu \Phi + m^2 \Phi = 0$$

Substitute

$$\phi(x) = f_k(\eta) e^{ikx}$$

then eom is

$$f_k''(\eta) + (k^2 + a^2(\eta) m^2) f_k(\eta) = 0.$$

Mathematica can solve this equation! The answer can be written in terms of hypergeometric functions

The solutions are

$$u_k^{\text{in}}(\eta, x) = (4\pi\omega_{\text{in}})^{-\frac{1}{2}} \exp\{ikx - i\omega_+\eta - (i\omega_-/\rho)\ln[2\cosh(\rho\eta)]\}$$

$$\times\, {}_2F_1(1 + (i\omega_-/\rho), i\omega_-/\rho; 1 - (i\omega_{\text{in}}/\rho); \tfrac{1}{2}(1 + \tanh\rho\eta))$$

$$\xrightarrow[\eta \to -\infty]{} (4\pi\omega_{\text{in}})^{-\frac{1}{2}} e^{ikx - i\omega_{\text{in}}\eta},$$

where

$$\left.\begin{aligned} \omega_{\text{in}} &= [k^2 + m^2(A - B)]^{\frac{1}{2}} \\ \omega_{\text{out}} &= [k^2 + m^2(A + B)]^{\frac{1}{2}} \\ \omega_\pm &= \tfrac{1}{2}(\omega_{\text{out}} \pm \omega_{\text{in}}). \end{aligned}\right\}$$

The solutions in the future are

$$u_k^{\text{out}}(\eta, x) = (4\pi\omega_{\text{out}})^{-\frac{1}{2}} \exp\{ikx - i\omega_+\eta - (i\omega_-/\rho)\ln[2\cosh(\rho\eta)]\}$$
$$\times\ {}_2F_1(1 + (i\omega_-/\rho), i\omega_-/\rho; 1 + (i\omega_{\text{out}}/\rho); \tfrac{1}{2}(1 - \tanh\rho\eta))$$
$$\xrightarrow[\eta \to +\infty]{} (4\pi\omega_{\text{out}})^{-\frac{1}{2}} e^{ikx - i\omega_{\text{out}}\eta}.$$

These are related via hypergeometric transformations

we can write

$$u^{in} = \alpha_k u_k^{out} + \beta_k (u_{-k}^{out})^*$$

with $$\alpha_R = \left(\frac{\omega_{out}}{\omega_{in}}\right)^{1/2} \frac{\Gamma\left(1-\frac{i\omega_{in}}{\rho}\right)\Gamma\left(-i\omega_{out}/\rho\right)}{\Gamma\left(-i\omega_+/\rho\right)\Gamma\left(1-i\omega_+/\rho\right)}$$

$$\beta_R = \left(\frac{\omega_{out}}{\omega_{in}}\right)^{1/2} \frac{\Gamma\left(1-i\omega_{in}/\rho\right)\Gamma\left(i\omega_{out}/\rho\right)}{\Gamma\left(i\omega_-/\rho\right)\Gamma\left(1+(i\omega_-/\rho)\right)}$$

As we discussed, many of the novel features of QFT in curved spacetime also arise in flat space. So we turn to flat space in 2 dimensions and quantize it in 2 different ways.

Consider coordinates t, x so that the metric is

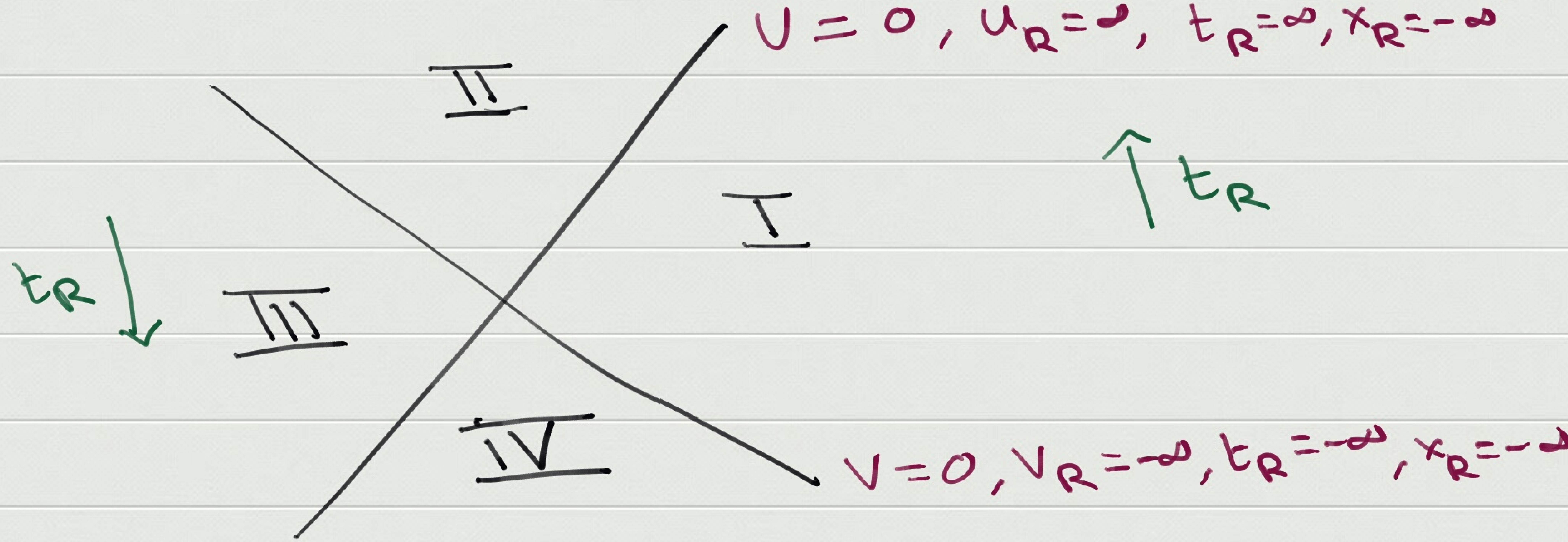
$$ds^2 = dt^2 - dx^2 = dU\,dV \quad ; \quad U = t - x, \quad V = t + x$$

We now write

$$U = -\frac{1}{a} e^{-aU_r}$$

$$V = \frac{1}{a} e^{aV_r}$$

in quadrant I.



Notice that $\displaystyle dU\,dV = \frac{1}{a^2}\, e^{a(v_R-u_R)}\, du_R\, dv_R.$

$$= \frac{1}{a^2}\, e^{2ax_R}\left[dt_R^2 - dx_R^2\right]. \quad , \quad \begin{aligned} u_R &= t_R - x_R \\ v_R &= t_R + x_R \end{aligned}$$

Consider an observer moving under a constant force.

We have

$$\frac{d}{dt}\left[\left(\frac{dx}{dt}\right) \Big/ \sqrt{1-\left(\frac{dx}{dt}\right)^2}\,\right] = F/m = a$$

with $dx/dt = 0$ at $t=0$, this is solved by

$$x = \sqrt{\frac{1}{a^2} + t^2} + c$$

constant

with $c=0$, we see the eqn of the worldline is

$$x^2 - t^2 = \frac{1}{a^2}$$

or $UV = \frac{-1}{a^2}$ or $\boxed{x_R = 0}$

So $x_R = 0$ is the trajectory of a constantly accelerated observer. This observer's proper time is

$$\tau = \int \sqrt{1-\dot{x}(t)^2}\, dt = \frac{1}{a}\sinh^{-1}(at)$$

or

$$\frac{1}{2a}\left(e^{a\tau} - e^{-a\tau}\right) = t$$

but we also have $t = \frac{U+V}{2} = \frac{1}{2a}\left(e^{av_R} - e^{-au_R}\right)$

At $x_R = 0$ this means

$$t = \frac{1}{2a}\left(e^{at_R} - e^{-at_R}\right)$$

so

$t_R = \tau$ = proper time of this accelerated observer

Now, note that

$$\sqrt{-g} = e^{2ax_R}/a^2 = \left(g^{t_R t_R}\right)^{-1} = \left(g^{x_R x_R}\right)^{-1}.$$

So the wave equation in these coordinates is again

$$\left(\frac{\partial^2}{\partial t_R^2} - \frac{\partial^2}{\partial x_R^2}\right)\phi = 0$$

$$\Rightarrow \phi_{I} = \int \frac{d\omega}{\sqrt{\omega}}\left[e^{-i\omega u_R} a_\omega + e^{-i\omega v_R} b_\omega + h.c\right]$$

Similarly, in region $\overline{\underline{III}}$ we introduce a second set of Rindler coordinates through

$$U = \frac{1}{a} e^{-au_R} \; ; \; V = \frac{-1}{a} e^{av_R}$$

The field here looks like

Note +ve signs because of direction of t_R

$$\Phi_{\text{III}} = \int \frac{d\omega}{\sqrt{\omega}} \left[\tilde{a}_\omega e^{i\omega u_R} + \tilde{b}_\omega e^{i\omega v_R} + h.c. \right]$$

If we think of the slice at $T=0$, we can then write

$$\Phi = \int \frac{d\omega}{\sqrt{\omega}} \Big[\tilde{a}_\omega U_L(u_r) + \tilde{b}_\omega V_L(V_R) + a_\omega U_R(u_R) + b_\omega V_R(V_R) + h.c. \Big]$$

where

$$V_L(u_R) = e^{i\omega u_R}, \quad \text{Region III (on left)}$$
$$= 0, \quad \text{region I (on right)}$$

$$V_R(u_R) = 0, \quad \text{Region III}$$
$$e^{-i\omega u_R}, \quad \text{Region I}$$

$$V_L(v_R) = e^{i\omega v_R}, \quad \overline{\text{III}}$$
$$= 0, \quad \overline{\text{I}}$$

$$V_R(v_R) = 0, \quad \overline{\text{III}}$$
$$e^{-i\omega v_R} \quad \overline{\text{I}}$$

On the other hand, we can also write

$$\Phi = \int \frac{d\omega}{\sqrt{\omega}} \left[c_\omega e^{-i\omega(t-x)} + d_\omega e^{-i\omega(t+x)} + h.c. \right]$$

ordinary Minkowski coordinates

What are the Bogoliubov coefficients between these two expansions?