QABH- Lecture 6

Last time, we derived a formula for the light-cone limit of two-point functions in d-dimensions.

$$\lim_{U_1 - U_2 \to 0} \langle \partial_{V_1} \Phi(U_1, V_1, x_1)\, \partial_{V_2} \Phi(U_2, V_2, x_2) \rangle = \propto \frac{\delta^{d-1}(x_1 - x_2)}{(V_1 - V_2)^2}.$$

This result is independent of the masses and details of the self-interaction of the field.

We will now rederive this result using Rindler modes.
Recall that the Rindler modes were

Recall that the Rindler modes were given by

$$\phi(U_1, V, x) = \int \frac{dw}{\sqrt{w}} \, a_w e^{ik \cdot x_R} e^{-iwt_R} \frac{K_{iw}(e^{x_R}|k|)}{\Gamma(iw)} + h.c.$$

in the limit where $x_R \to -\infty$ or $U \to 0$, this becomes

$$\phi(U_1, V, x) \to \int \frac{dw}{\sqrt{w}} \, a_{w,R} e^{ik \cdot x_R} e^{-iwt_R} \left(e^{i\delta} e^{iwx_R} + e^{-iwx_R} e^{-i\delta} \right)$$

Note that we can write this as

$$\phi \xrightarrow[v\to 0]{} \int \frac{d\omega}{\sqrt{\omega}} a_{\omega,k} e^{ik\cdot x_1} \left[e^{i\delta} v^{i\omega} + e^{-i\delta} V^{-i\omega} \right] + h.c.$$

If we consider the correlator in question, we find

$$\langle \partial_{v_1}\phi(v_1, V, x_1)\partial_{v_2}\phi(v_2, V_2, x_2)\rangle .$$

from derivative

$$= \int d^{d-1}k \frac{d\omega}{\sqrt{\omega}} \frac{d\omega'}{\sqrt{\omega'}} \Big(e^{i(kx_1 - k'x_2)} \left[\langle a_{\omega,k} a^{+}_{\omega',k'} \rangle \omega\omega' V_1^{-i\omega} V_2^{i\omega'} \right]$$

$$e^{-ikx_1 + k'x_2} \left[\langle a^{+}_{\omega,k} a_{\omega',k} \rangle (\omega\omega') V_1^{i\omega} V_2^{-i\omega'} \right] \Big) \times \frac{1}{V_1 V_2}$$

from derivative

We have made several simplifications here. We have set

$$\langle a_{\omega, R} \, a_{\omega', R'} \rangle = 0 = \langle a^{\dagger}_{\omega, R} \, a^{\dagger}_{\omega' R'} \rangle$$

This is because we have assumed that correlators in the Minkowski vacuum are invariant under Rindler time-translations

This follows from the geometry. In Rindler coordinates, the metric is

$$ds^2 = e^{2x_R} (dt_R^2 - dx_R^2) + dx_{d-1}^2$$

which is left invariant by

$$t_R \to t_R + \tau$$

so

$$\langle \Omega_M | a_{\omega,R} \, a_{\omega',R'} | \Omega_M \rangle = \langle \Omega_M | e^{iH_R \tau} a_{\omega,R} \, a_{\omega',R'} e^{-iH_R \tau} | \Omega_M \rangle$$

$$= e^{-i(\omega+\omega')\tau} \langle \Omega_M | a_{\omega,R} \, a_{\omega',R'} | \Omega_M \rangle$$

Since $\omega, \omega' > 0$ this implies that $\langle \Omega_M | a_{\omega,R} \, a_{\omega',R'} | \Omega_M \rangle = 0$.
We are not stating that $H_R |\Omega_M\rangle = 0$
but rather just using invariance of correlators.

By the same arguments

$$\langle a^\dagger_{\omega,k}\, a^\dagger_{\omega',k'} \rangle = 0$$

and

$$\langle a_{\omega,k}\, a^\dagger_{\omega',k'} \rangle = c(k,\omega)\, \delta(k'-k)\,\delta(\omega'-\omega)$$

$$\langle a^\dagger_{\omega,k}\, a_{\omega,k'} \rangle = [c(k,\omega)-1]\, \delta(k'-k)\,\delta(\omega'-\omega)$$

where c is a function we need to determine.

Now we notice that due to ultralocality, c must be independent of k.

Putting this in, we find that

$$\langle \partial_{v_1} \phi(u_1, v_1, x_1)\, \partial_{v_2} \phi(u_2, v_2, x_2) \rangle$$

$$= \delta^{d-1}(x_1 - x_2) \int_0^\infty d\omega\, \omega \left[C(\omega) \left(\frac{v_2}{v_1}\right)^{i\omega} + (C(\omega)-1)\left(\frac{v_1}{v_2}\right)^{i\omega} \right]$$

So we have recovered the ultra-locality.

Now, we will verify that

$$C(\omega) = \frac{1}{1-e^{-\beta\omega}}$$

with $\beta = 2\pi$

is the right answer.

Note that $C(\omega)-1 = \dfrac{e^{-\beta\omega}}{1-e^{-\beta\omega}}$

So our integral is

$$\int_0^\infty d\omega\ \omega \left[\frac{e^{-\beta\omega}}{1-e^{-\beta\omega}} \left(\frac{V_2}{V_1}\right)^{i\omega} + \frac{1}{1-e^{-\beta\omega}} \left(\frac{V_1}{V_2}\right)^{i\omega} \right]$$

But note that

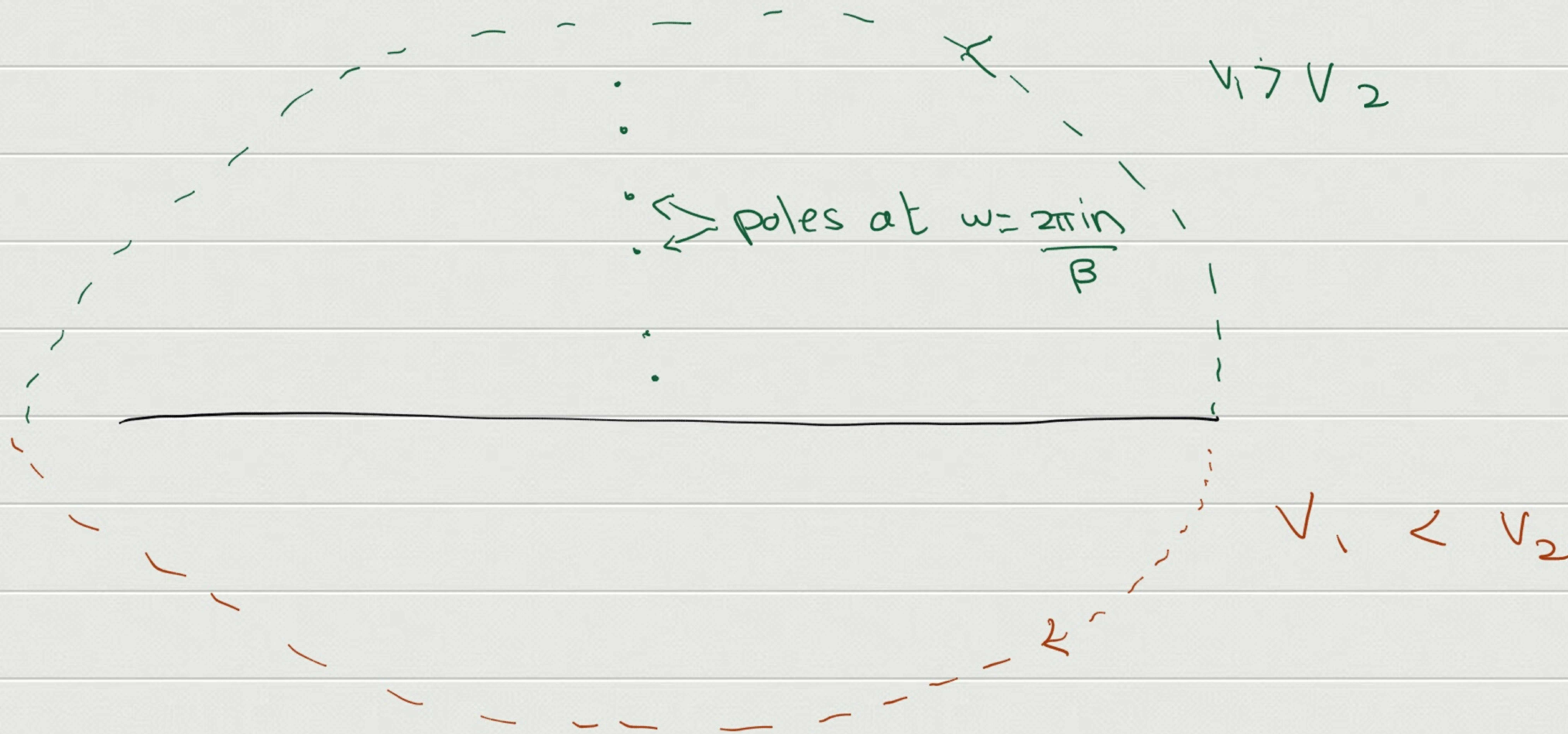
$$C(-\omega) = \frac{1}{1-e^{\beta\omega}} = -\left[C(\omega)-1\right]$$

Taking into account the additional minus sign from ω, we see the integral can be converted to $(-\infty, \infty)$

$$\int_{-\infty}^{\infty} d\omega \, \frac{\omega}{1-e^{-\beta\omega}} \left(\frac{V_1}{V_2}\right)^{i\omega}$$

If $V_1 < V_2$, then we can complete the contour in the lower half plane; if $V_1 > V_2$ complete in the upper half plane.
poles of the function are at

$$\beta\,\omega = 2\pi i n$$



$V_1 > V_2$
poles at $\omega = \frac{2\pi i n}{\beta}$
$V_1 < V_2$

Picking up residues from the poles, we find:

$$\int_{-\infty}^{\infty} \frac{\omega}{1-e^{-\beta\omega}} \left(\frac{V_1}{V_2}\right)^{i\omega} = \sum_{n>0} \frac{2\pi i n}{\beta} \left(\frac{V_1}{V_2}\right)^{-2\pi n/\beta}.$$

Recall that

$$\sum n x^n = \frac{x}{(1-x)^2}$$

can see from

$$\sum n x^n = x \frac{\partial}{\partial x} \sum x^n = \frac{x}{(1-x)^2}$$

So the integral gives us

$$\frac{\left(\frac{V_1}{V_2}\right)^{2\pi/\beta}}{\left(1-\left(\frac{V_1}{V_2}\right)^{2\pi/\beta}\right)^2} = \frac{(V_1 V_2)^{2\pi/\beta}}{\left(\delta V^{2\pi/\beta}\right)^2}$$

Now it is clear that we need $\beta = 2\pi$.
With this, and combining other factors we find again

$$\lim_{U_1-U_2\to 0} \langle \partial_{V_1}\phi(U_1,V_1,X_1)\,\partial_{V_2}\phi(U_2,V_2,X_2)\rangle \propto \frac{\delta^{d-1}(x_1-x_2)}{\delta V^2} \;!$$

This is a very useful derivation since it just relies on local properties of correlators. as opposed to relying on global properties of modes.

To complete our discussion of QFT in curved space, we describe Unruh detectors [Birrell & Davies 3.3]

A lot of discussions of QFT in curved space focus on this, but we have de-emphasized it a little to emphasize that correlation functions are more useful as probes.

To model a particle detector, we turn on a coupling along the worldline of a moving observer

$$H_{int} = \alpha J(t) \Phi(x(t))$$

where $x(t)$ specifies the position at proper time t.
If detector starts in some state $|0\rangle$, the amplitude that it will transition to an excited state $|e\rangle$ is:

$$M = \int \langle e, \psi | H_{int} | 0, x \rangle . dt$$

where $|x\rangle, |\psi\rangle$ are the initial & final states of the field.
The total probability that the detector will transition to $|e\rangle$ is given by squaring and

throwing away information about ψ. This is

$$\int |M|^2 d\psi = \int \langle 0, x | H_{int}(t') | e, \psi \rangle \langle e, \psi | H_{int}(t) | 0, x \rangle \, dt \, dt'$$

$$= \alpha^2 \int \langle 0 | J(t') | e \rangle \langle e | J(t) | 0 \rangle \langle x | \phi(x(t')) \phi(x(t)) | x \rangle \cdot dt \, dt'$$

$$\propto \int \langle x | \phi(t', x(t')) \phi(t, x(t)) | x \rangle \, e^{iE(t-t')} \, dt \, dt'$$

where E is the energy diff between $|e\rangle$ and $|0\rangle$ and the constant of proportionality is

$$\alpha^2 |\langle 0 | J(0) | e \rangle|^2 .$$

So the probability of the detector clicking is an integral of the 2-pt function over the particle trajectory. The 2-pt function that appears here is the Wightman function. $G(x, t)$

Recall that for a 4-dim scalar field the time-ordered Greens function is given by

$$G_T = \langle T \phi(x, t) \phi(0,0) \rangle = \frac{1}{(x^2 - t^2 + i\varepsilon)}$$

For $t > 0$, $G = G_T \qquad \frac{1}{x^2 - t^2 + i\varepsilon}, \quad t > 0$

For $t < 0 \quad G = G_T^* = \frac{1}{x^2 - t^2 - i\varepsilon}, \quad t < 0$

This can be combined into

$$G = \frac{1}{x^2 - (t - i\varepsilon)^2}$$

As an application, we show a detector on a uniformly accelerated trajectory sees a thermal path of particles.

Recall that the trajectory of this particle is

$$z = \frac{1}{a}\cosh(a\tau) \; ; \quad t = \frac{1}{a}\sinh(a\tau)$$

So the integral we need to evaluate is

$$a^2 \int \frac{d\tau\, d\tau' \; e^{iE(\tau-\tau')}}{(\cosh a\tau - \cosh a\tau')^2 - (\sinh a\tau - \sinh a\tau' - i\varepsilon)^2}$$

$$= 2a^2 \int \frac{d\tau\, d\tau' \, e^{iE(\tau-\tau')}}{\left[\sinh\left(a\frac{(\tau-\tau')}{2} + i\varepsilon\right)\right]^2}$$

The simplification above arises by expanding the bracket and then recognizing that the coeff of $i\varepsilon$ is +ve for $\tau > \tau'$

This integral is divergent! But the probability per unit time is finite and given by

$$2a^2 \int \frac{e^{-iE\tau}}{\left[\sinh \frac{a\tau}{2} - i\varepsilon\right]^2} d\tau$$

We can complete the contour through the lower half plane. poles are at $\frac{a\tau}{2} = i\pi n$ with residue given by

$$16\pi E e^{-2\pi E n/a}.$$

The pole at $n=0$ is excluded.

So we find the net probability per unit time is proportional to

$$\frac{dP}{dt} \propto \frac{E\, e^{-2\pi E/a}}{1 - e^{-2\pi E/a}} = \frac{E}{e^{2\pi E/a} - 1}$$

So once again we see that the probability of transition per unit time looks thermal with

$$\beta = \frac{2\pi}{a}.$$