# Supersymmetric Partition Functions in the AdS/CFT Conjecture

A dissertation presented

by

Suvrat Raju

to

The Department of Physics

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in the subject of

Physics

Harvard University

Cambridge, Massachusetts

June 2008

Thesis advisor | Author

**Shiraz Minwalla** | **Suvrat Raju**

**Supersymmetric Partition Functions in the AdS/CFT Conjecture**

# Abstract

We study supersymmetric partition functions in several versions of the AdS/CFT correspondence.

We present an Index for superconformal field theories in $d = 3, 4, 5, 6$. This captures all information about the spectrum that is protected, under continuous deformations of the theory, purely by group theory. We compute our Index in $\mathcal{N} = 4$ SYM at weak coupling using gauge theory and at strong coupling using supergravity and find perfect agreement at large $N$. We also compute this Index for supergravity on $\text{AdS}_4 \times S^7$ and $\text{AdS}_7 \times S^4$ and for the recently constructed Chern Simons matter theories.

We count 1/16 BPS states in the free gauge theory and find qualitative agreement with the entropy of big black holes in $\text{AdS}_5$. We note that the near horizon geometry of some small supersymmetric black holes is an extremal BTZ black holes fibered on a compact base and propose a possible explanation for this, based on giant gravitons. We also find the partition function of the chiral ring of the $\mathcal{N} = 4$ SYM theory at finite coupling and finite $N$.

Turning to $\text{AdS}_3$, we study the low energy 1/4 and 1/2 BPS partition functions by finding all classical supersymmetric probe brane solutions of string theory on this background. If the background $B_{NS}$ field and theta angle vanish, $\mathrm{AdS}_3 \times S^3 \times T^4/K3$ supports supersymmetric probes: D1 branes, D5 branes and bound states of D5 and D1 branes. In global AdS, upon quantization, these solutions give rise to states in discrete representations of the SL(2,R) WZW model on $\mathrm{AdS}_3$.

We conclude that (a) the 1/4 BPS partition function jumps if we turn on a theta angle or NS-NS field (b) generic 1/2 BPS states are protected. We successfully compare our 1/2 BPS partition function with that of the symmetric product. We also discuss puzzles, and their possible resolutions, in reproducing the elliptic genus of the symmetric product.

Finally, we comment on the spectrum of particles in the theory of gravity dual to *non-supersymmetric* Yang Mills theory on $S^3\times$ time.

# Contents


Previously Published Work . . . vii
Acknowledgments . . . viii

**1 Introduction** **1**
1.1 Background . . . 1
1.2 Motivation . . . 4
1.3 Summary of Results . . . 5

**2 An Index for 4 Dimensional Superconformal Field Theories** **10**
2.1 Introduction . . . 10
2.2 Unitary Representations of 4 dimensional Superconformal Algebras . 13
2.3 A Trace Formula for the Index . . . 25
2.4 Computation of the Index in $\mathcal{N} = 4$ Yang Mills on $S^3$ . . . 31
2.5 Discussion . . . 39

**3 Supersymmetric Partition Functions in $\mathbf{AdS_5/CFT_4}$** **41**
3.1 Introduction . . . 41
3.2 The partition function over $\frac{1}{16}$ BPS states . . . 43
3.3 Small Supersymmetric Black Holes in $AdS_5$ as Giant Gravitons . . . 53
3.4 Near Horizon Geometry of Supersymmetric Black Holes in $\mathrm{AdS}_5$ . . . 56
3.5 Partition Functions over $\frac{1}{2}$, $\frac{1}{4}^{th}$ and $\frac{1}{8}^{th}$ BPS States . . . 66
3.6 Comparing the Cohomological Partition Function and the Index . . . 74
3.7 Discussion . . . 77

**4 Indices for Superconformal Field Theories in $3, 5$ and $6$ Dimensions** **79**
4.1 Introduction . . . 79
4.2 d=3 . . . 81
4.3 d=6 . . . 105
4.4 d=5 . . . 127
4.5 Discussion . . . 135

**5 Supersymmetric States in $\mathrm{AdS}_3/\mathrm{CFT}_2$ I : Classical Analysis 137**
5.1 Introduction . . . 137
5.2 Killing spinor and kappa symmetry analysis . . . 141
5.3 Charge Analysis: D strings . . . 159
5.4 Charge Analysis: D1-D5 bound state probes . . . 172
5.5 Moving off the Special Point in Moduli Space . . . 184
5.6 Semi-Classical Quantization . . . 191
5.7 Results and Discussion . . . 197
5.A Miscellaneous Technical Details . . . 199

**6 Supersymmetric States in $\mathrm{AdS}_3/\mathrm{CFT}_2$ II : Quantum Analysis 202**
6.1 Introduction . . . 202
6.2 Classical Supersymmetric Solutions in Global $AdS_3$ . . . 208
6.3 Another Approach to Classical Solutions:
'Polyakov' Action . . . 218
6.4 Semi-Classical Quantization . . . 230
6.5 Exact Analysis of the D-string . . . 246
6.6 Results . . . 258
6.A Technical Details of the Spacetime Partition Function . . . 260
6.B Theta Functions . . . 267

**7 Future Directions 268**

**A Moving Away from Supersymmetry 270**
A.1 Introduction . . . 270
A.2 Conformal Algebra . . . 271
A.3 Representations of the Conformal Group . . . 274
A.4 Characters . . . 276
A.5 Oscillator Construction . . . 278
A.6 Character Decomposition . . . 279
A.7 The Integral . . . 282
A.8 Proof of Integral Multiplicities . . . 288
A.9 $\mathcal{N} = 4$ SYM . . . 290
A.10 Summary . . . 291

**Bibliography 292**

# Previously Published Work

1. Chapters 1 and 2 of this thesis are based largely on the paper

   **"An Index for 4 dimensional super conformal theories"**
   J. Kinney, J. M. Maldacena, S. Minwalla and S. Raju
   Commun. Math. Phys. **275**, 209 (2007) [arXiv:hep-th/0510251]

2. In addition, Chapter 2 also includes material from the unpublished work

   **"Near Horizon Geometry of Extremal Black Holes in $AdS_5$"**
   S. Kim, S. Minwalla, S. Raju, S. Trivedi

3. Chapter 3 is based on the paper

   **"Indices for Superconformal Field Theories in 3,5 and 6 Dimensions"**
   J. Bhattacharya, S. Bhattacharyya, S. Minwalla and S. Raju
   JHEP **0802**, 064 (2008) [arXiv:0801.1435 [hep-th]]

4. Chapter 4 is based on the paper

   **"Supersymmetric Giant Graviton Solutions in $AdS_3$"**
   G. Mandal, S. Raju and M. Smedback
   Phys. Rev. D **77**, 046011 (2008) [arXiv:0709.1168 [hep-th]]

5. Chapter 5 is based on the paper

   **"Counting Giant Gravitons in $AdS_3$"**
   S. Raju
   Phys. Rev. D **77**, 046012 (2008) [arXiv:0709.1171 [hep-th]]

6. Appendix A is based on the paper

   **"The spectrum of Yang Mills on a sphere"**
   A. Barabanschikov, L. Grant, L. L. Huang and S. Raju
   JHEP **0601**, 160 (2006) [arXiv:hep-th/0501063]

# Acknowledgments

First, I would like to thank my advisor Shiraz Minwalla for his advice and encouragement throughout my PhD work.

The physics department at Harvard provided a very vibrant intellectual atmosphere for my studies. I benefited tremendously from interactions with my fellow graduate students and from many discussions about physics, philosophy, politics and everything else in the universe and beyond it! I would especially like to thank my many friends and colleagues in the physics department – Lars Grant, Kyriakos Papadodimas, Joe Marsano, Subhaneil Lahiri, Monica Guica, Mohammad Hafezi, Philippe Larochelle, Morten Ernebjerg, Itay Yavin, Josh Lapan, Kirill Saraikin, Xi Yin, Georgios Pastras, Jesse Thaler, Mboyo Esole, Clay Cordova, David Simmons-Duffin, Mason Klein, Yina Mo, Christine Wang, Param Dhillon, Wei Li, Damon Clark, Michael Hohensee, Daniel Babich and John Paul Chou. I would like to thank Jihye Seo for comments on a draft on this thesis. Lubos Motl and Nima Arkani-Hamed and later Frederik Denef ensured that the fourth and fifth floors of Jefferson were always lively and exciting. I would like to thank Peter Galison for introducing me to the History of Science group at Harvard and for several educative discussions.

Although my interaction with Sidney Coleman was limited, my understanding of quantum field theory is largely due to him.

I would like to acknowledge the administrative staff at Harvard, particularly Sheila Ferguson and Dayle Maynard, for their patient help with everything.

I spent part of my PhD at TIFR and I would like to thank Rudra Jena, Rahul Nigam, Suresh Nampuri, R. Loganayagam, Anindya Mukherjee, Pallab Basu, Ashiq Iqbal, Argha Banerjee, Sayantani Bhattacharyya, Jyotirmoy Bhattacharya, Apoorva

Nagar and everyone else in the theory student room. I would also like to thank K. Narayan, Kevin Goldstein, Sandip Trivedi and Gautam Mandal for many discussions.

I would like to acknowledge my many friends outside the the physics department, particularly Dilini Pinnaduwage, Antara Datta, Kumar Sambhav and Shruti Mohanty. Manasi was a source of constant support and even made the Mumbai local trains bearable.

An extremely important influence on me, in my years as a graduate student, came from my fellow activists. I would like to acknowledge everyone in Boston, who worked with me as part of the Harvard Initiative for Peace and Justice, the Socialist Alternative, the Boston Stop the Wars Coalition, the Association for India's Development, the Alliance for a Secular and Democratic South Asia and particularly Kaveri who was a constant comrade.

In India I would like to acknowledge the members of the Young Activist Movement, the Lokshahi Hakka Sangathana, the Research Unit in Political Economy, the National Alliance for Peoples Movements in Mumbai and the TIFR Education Group.

I was shaped by my family and to them I dedicate this thesis.

*To Ma, Pa and Auchu.*

# Chapter 1

# Introduction

## 1.1 Background

Physical theories are judged by both internal and external consistency. We look for theories that are logically consistent and describe the world accurately. Both these principles were important in the $20^{th}$ century. For example, historically, Dirac was led to the celebrated 'Dirac equation' because he was troubled by the negative probabilities that appear in the Klein Gordon equation. On the other hand what is today called the standard model of particle physics was chosen over other logical alternatives because it seemed to provide a more accurate description of the world.

In this context, the study of quantum gravity marks a peculiar epoch in theoretical physics. The standard model – which is based on the framework of quantum field theory – and classical general relativity are consistent with all experiments that have been performed to date. However, it has so far proven to be impossible to incorporate gravity in the framework of traditional quantum field theory. The story of quantum gravity is entirely about this attempt: theoretical physicists are looking for *any* theory that reduces to general relativity in the classical limit and is consistent with the principles of quantum mechanics!

There are two factors that prevent the union of gravity and quantum mechanics. First, when gravity is weak and approximately linear, classical General Relativity predicts the existence of 'gravity waves'. Just like light waves are described by photons we expect, that in quantum mechanics, gravity waves should be described by 'gravitons'. However, the interactions of these gravitons with themselves or with other particles are necessarily 'non-renormalizable'. Non-renormalizable theories are sometimes used as 'effective theories'; however, beyond some energy scale, they need to be supplemented with additional structure (this is called 'UV-completing' the theory). UV-completions are seldom unique. To the contrary most non-renormalizable theories, such as the Fermi theory of weak interactions, allow an infinite choice of UV-completions. Gravity is strikingly different; so far no one has found even one UV completion of gravity within the framework of quantum field theory.

Second, in the opposite regime from gravity waves, when gravity is highly non-linear, we find that general relativity predicts the existence of black holes. Black holes have an event horizon; something that crosses this horizon never returns. However, in the seventies, it was found, by applying the principles of quantum field theory to black holes, that they must emit radiation. This radiation is exactly thermal and causes the black hole to evaporate slowly. When the black hole has evaporated completely we end up with thermal radiation but no trace of what constituted the black hole in the first place. This loss of information is in contradiction with the unitarity of quantum evolution. Once again, we find that trying to put quantum mechanics together with gravity leads to a paradox.

So we would like a theory of quantum gravity that describes quantum interactions between gravitons and explains how unitarity is preserved in Hawking radiation. If possible we would also like this theory to provide a 'microscopic' understanding of black hole thermodynamics. At the outset one may imagine that it is easy to conjure models that meet these requirements. However, this task has proven remarkably difficult. The great virtue of string theory is that it meets specifications that are very close to those listed above.

The best example of how string theory answers fundamental questions about gravity comes from the AdS/CFT correspondence. This remarkable conjecture goes as follows. We consider a special kind of gauge theory – $\mathcal{N} = 4, SU(N)$ Yang Mills theory. This is an ordinary gauge theory, albeit one with a lot of symmetry. According to the AdS/CFT conjecture, at strong coupling, this theory is identical to a theory of gravity coupled to some special matter, living in Anti-de Sitter space! We know how to quantize the gauge theory. Hence, if the AdS/CFT conjecture is correct it provides us with a theory of quantum gravity.

This theory of quantum gravity is a close cousin of ordinary four dimensional gravity. For example, at low energies, this theory has excitations that are gravitons and we will study these excitations in the second chapter of this thesis. This theory also has black holes and we will study these in the third chapter. At energies between those of gravitons and black holes, this theory has some special kinds of stringy states and we will study these in the fourth and fifth chapters of this thesis.

In the description above, we used a gauge theory to describe a specific theory of quantum gravity. Logically, this is entirely consistent. However, we also have an independent definition of this theory of quantum gravity via string theory. So, at times we can use the string theory to perform calculations that are difficult in the gauge theory. We will use both these perspectives in this thesis.

This sets the backdrop for this work. We will now describe the motivation for the problems that we solve in some more detail and using a more technical tone.

## 1.2 Motivation

As we explained above, the AdS/CFT conjecture links gravity on Anti-de sitter space with a conformal field theory. There are several versions of this correspondence. In the best studied example, the theory of gravity is Type IIB string theory on $\text{AdS}_5 \times S^5$ and the conformal field theory is $\mathcal{N} = 4, SU(N)$ Yang Mills theory living on the boundary of $\text{AdS}_5$. The coupling constant of the conformal field theory is related to the string coupling constant in gravity: $g^2_{\text{YM}} \sim g_s$. The radius of Anti-de Sitter space, in string units, is related to the 't Hooft coupling: $\left(\frac{R}{l_s}\right)^4 \sim \lambda = g^2_{\text{YM}} N$. A precise way to state the AdS/CFT conjecture is in terms of the partition functions for each theory: $Z = \text{tr} e^{\beta H}$, where $H$ is the Hamiltonian. The AdS/CFT conjecture then tells us

$$Z_{\text{gauge theory}}(\beta, g^2_{\text{YM}}, N) = Z_{\text{string theory}}(\beta, g_s = g^2_{YM}, \frac{R}{l_s} = \left(g^2_{\text{YM}} N\right)^{\frac{1}{4}}) \tag{1.1}$$

However equation (1.1) is very hard to verify. We see, from the formulae above, that when Yang Mills perturbation theory is valid and the 't Hooft coupling is small, the radius of AdS space is small in string units. This means that stringy corrections are important in gravity. On the other hand, when the radius of AdS space is large, the Yang Mills coupling is large and we cannot use perturbation theory. Thus we see that while we have a very interesting conjectured duality, we also have a conundrum: the string theory is easy to analyze when the gauge theory is not and vice-versa.

This thesis concerns itself with a simplified version of (1.1). We will consider, on each side of the duality, not the full partition functions but the *supersymmetric* partition functions. A supersymmetric partition function is defined as a trace over *supersymmetric* states only: $Z_{\text{susy}} = \text{tr}_{\text{susy}} e^{\beta H}$. We can now consider a simplified version of (1.1). Namely

$$Z_{\text{susy, gauge theory}}(\beta, g^2_{\text{YM}}, N) = Z_{\text{susy, string theory}}(\beta, g_s = g^2_{YM}, \frac{R}{l_s} = \left(g^2_{\text{YM}} N\right)^{\frac{1}{4}}). \quad (1.2)$$

Remarkably (1.2) turns out to be tractable and in several cases we will be able to calculate both sides of the equation above and demonstrate that they are equal.

We now turn to a summary of the results that we have obtained.

## 1.3 Summary of Results

Supersymmetric partition functions are often protected under deformations of a theory. Often we need to study the dynamics of the theory to argue this. However, sometimes group theory alone is enough to guarantee that some quantities cannot change under a class of deformations; in this thesis, we will use the word 'Index' to refer to such quantities.

In chapter 2, we describe how to construct such indices in four dimensional superconformal field theories. The idea behind this construction is as follows. Some representations of the superconformal algebra are short i.e. they have fewer states than generic representations. The energy of the highest weight in this representation is determined in terms of its other charges; this is called a BPS relation. Under continuous deformations of a theory, these charges cannot change since they are integrally quantized. One might be tempted to think that the energy also cannot change and that all quantum numbers of this representation are protected under deformations; this is not quite correct.

The is because two or more short representations can combine to form long representations. Group theory cannot predict when such a combination occurs. However, it does tell us that such a combination can only occur between a bosonic and a fermionic representation. Thus, the *difference*

$$\text{number of bosonic short reps} - \text{number of fermionic short reps},$$

is protected! The quantity above is the simplest example of an Index and was first studied by Witten. The indices that we define in Chapter 2, are generalizations of this construction. We proceed to apply our construction to the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence. We calculate the superconformal Index in the free gauge theory and also using supergravity and find perfect agreement.

However, it is evident that this Index does not capture all information about supersymmetric states in the theory. For example, the theory of gravity has supersymmetric black holes and these do not contribute to the Index at all (the contribution from bosonic and fermionic states that constitute the black hole cancels out). In chapter 3, we turn to more general supersymmetric partition functions in $\mathrm{AdS}_5/\mathrm{CFT}_4$. We calculate the partition function over states that preserve $\frac{1}{16}$ of the supersymmetry of the theory at free coupling. The entropy of such states shares qualitative features with the entropy of supersymmetric black holes. We conjecture that if one could count the number of $\frac{1}{16}$ supersymmetric states in the gauge theory at *any* finite coupling, this would reproduce the exact entropy of supersymmetric black holes.

Turning to states that preserve more supersymmetry, we calculate the partition function over $\frac{1}{8}$ BPS states at finite coupling and finite $N$. This partition function changes discontinuously if we turn on even a small finite coupling. However, we conjecture that it is protected under any further changes in the coupling. Non-trivial evidence for this conjecture has accumulated over the past few years and we will discuss this further in Chapter 3.

In Chaper 4, we extend the construction of our Index to superconformal field theory in 3,5 and 6 dimensions. We then calculate this Index in supergravity on $\mathrm{AdS}_4 \times S^7$ and $\mathrm{AdS}_7 \times S^4$. It is believed, that these supergravities are obtained as the large $N$, low energy limit of the theory of coincident $M2$ and $M5$ branes respectively. Although this is an area of active research, at this time, these latter theories are not understood independently. Any proposal for the worldvolume theories on $M2$ or $M5$ branes must reproduce, at large $N$, the indices we calculate in Chapter 4. In this chapter, we also apply our indices to Chern Simons matter theories. For such theories, we have the opposite problem. It is the gravity dual to these gauge theories that is not understood. Once again, any proposal for a gravity dual must reproduce the Index that we calculate.

In Chapters 6 and 5, we shift gears and turn to a study of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence. The Index in $AdS_3$ is quite famous and is called the 'elliptic genus'. This is the quantity that was first used to successfully count the entropy of black holes in string theory. Furthermore, almost every succesful analytic entropy counting programme, to date, may be phrased in terms of the $AdS_3/CFT_2$ correspondence.

However, the CFT involved in this correspondence is quite complicated. To be specific let us consider the duality between string theory on $AdS_3 \times S^3 \times T^4$ and a 2 dimensional conformal field theory. The CFT in questions is the $1+1$ dimensional sigma model on the moduli space of instantons in $\mathcal{N} = 4$ YM theory on $T^4$. This theory is not understood very well. However, it is believed that by tuning parameters in this theory, we can deform it to the sigma model on the symmetric product orbifold $(T^4)^N/S_N$, where $N$ is a parameter that controls the AdS radius. All calculations in the CFT are performed at this point in parameter space.

Several papers have tried to match supersymmetric partition functions calculated at this point in parameter space with the answers from supergravity. However, these papers have neglected the fact that supergravity itself ceases to be valid and string corrections are important after a certain energy.

In Chapter 4 we show that, for generic parameters in the CFT, this is justified and these string corrections are not important. However, at some singular points in parameter space, these string corrections can contribute to the supersymmetric partition function. To quantify this contribution, we use a technique that has been used frequently in the past few years. We start by enumerating all low energy supersymmetric *classical* solutions to string theory. This enumeration is performed in Chapter 5. In 6, we quantize these solutions, using techniques of geometric quantization, to obtain their contribution to the supersymmetric partition function. Our analysis leads to a prediction for new kinds of supersymmetric black hole solutions.

We are also left with the issue of understanding how the 'elliptic genus' jumps as we move off this singular point in moduli space. This is an important puzzle and may be useful in teaching us about strings in RR backgrounds. It may be the case, that the methodology we have used is not consistent and that would be interesting also.

To summarize then, we study supersymmetric partition functions in the AdS/CFT correspondence in diverse dimensions. We are often able to successfully match computations on both sides of parameter space but we are also left with puzzles for future work.

In the appendix, we include a brief note on moving away from supersymmetry. It is believed that large $N, SU(N)$ gauge theory should also have a string dual. In the appendix, we decompose the partition function of large $N$, $SU(N)$ gauge theory on a sphere to determine the spectrum of particles in the dual string theory. This is a check that the putative dual will have to satisfy.

# Chapter 2

# An Index for 4 Dimensional Superconformal Field Theories

## 2.1 Introduction

The best studied version of the AdS/CFT correspondence is between string theory on $AdS_5$ and a conformal field theory in 4 dimensions. This classic version of the correspondence was obtained by studying the infra-red limit of the theory on $N$ coincident $D3$ branes in flat space. An analysis of this system led to the conjecture that $\mathcal{N} = 4, U(N)$ Super Yang Mills theory is dual to type IIB string theory on $AdS_5 \times S^5$ [1].

The $\mathcal{N} = 4$ SYM theory, in the large $N$ limit, has a coupling constant $\lambda$, which is exactly marginal. This corresponds to the radius of $AdS$ space, in string units, with $\lambda \sim \left(\frac{R}{\alpha'}\right)^4$. We see now, that when supergravity is valid i.e when $\frac{R}{\alpha'} >> 1$, the gauge theory is strongly coupled and when Yang Mills perturbation theory is valid, string corrections are important in gravity.

Other versions of the AdS/CFT conjecture may be obtained by placing the D3 branes at orbifold singularities or at the tip of a conifold. In each of these cases, one obtains a situation as above, where Yang Mills perturbation theory can be trusted in one regime and gravity in another regime.

So, in order to check the AdS/CFT conjecture it is important to understand what quantities remain unchanged as one dials the coupling from weak to strong. These quantities can be computed both in gravity and in the perturbative gauge theory; if they match, this provides an important check of the AdS/CFT conjecture.

Such protected quantities, if they exist, generically fall into two classes. First, we have quantities that are protected by group theory alone. These quantities are called Indices. They do not change under any deformation of the spectrum that preserves the symmetries of the theory. For example, in $\mathcal{N} = 4$ Yang Mills theory, the Index is invariant not only under changes of the coupling constant, but under much more general deformations of the theory that preserve the the conformal symmetry and a particular supersymmetry.

Second, there are those quantities that are not 'Indices' in the sense above but are still invariant under changes of the coupling constant. We will discuss such supersymmetric partition functions in the next chapter.

Let us now discuss the principle behind the definition of an Index, in some more detail. Supersymmetry algebras that contain R-charges in the right hand side have special BPS multiplets. These multiplets occur at special values of energies or conformal dimensions determined by their charge, and have fewer states than the generic representation. An infinitesimal change in the energy of a special multiplet turns it into a generic multiplet with a discontinuously larger number of states. One might be tempted to use this observation to conclude that the number of short representations cannot change under variation of any continuous parameter of the field theory. However there is a caveat. It is sometimes possible for two or more BPS representations to combine into a generic representation. Such a combination, when it happens is always between a 'bosonic' BPS multiplet and a 'fermionic' BPS multiplet. So, the simplest Index one can define is the difference between the number of bosonic BPS multiplets and fermionic BPS multiplets.

If there is more symmetry in the problem, we can give more structure to our Indices. The Indices we will define below take the form

$$\mathcal{I}^W = Tr[(-1)^F e^{\mu_i q_i}], \tag{2.1}$$

and are defined for 4 dimensional superconformal field theories (with arbitrary number of supersymmetries) on $S^3\times$ time. The superscript 'W', is because these Indices closely resemble the Witten Index [2].

The Indices $\mathcal{I}^W$ are a functions of $2, 3$ and $4$ continuous variables for $\mathcal{N} = 1, 2, 4$ superconformal field theories respectively. We will explicitly computer this Index in the case of the $\mathcal{N} = 4$ Yang Mills theory in the free limit. We can also compute this Index in supergravity and on doing this computation, we find perfect agreement. This agreement provides a check on the AdS/CFT correspondence.

Now, it is known that $AdS_5 \times S^5$ supports supersymmetric black holes. Can our Index account for their entropy? Unfortunately, in the large $N$ limit, the Index does not undergo the deconfinement phase transition described in [3, 4]. In fact, as we mentioned above, the Index may be completely accounted for by 'supergravitons'. This indicates that the black holes completely cancel out in their contribution to the Index. We discuss this issue further in the next chapter.

The structure of this chapter is as follows. We start by discussing the superconformal algebra and its unitary representations. We show how superconformal Indices may be defined, given any such unitary representation. We then present a trace formula that may be used to evaluate all these Indices in a Lagrangian field theory. Finally, we compute this Index in $\mathcal{N} = 4$ SYM theory in the free limit and at strong coupling (using gravity) and find perfect agreement.

## 2.2 Unitary Representations of 4 dimensional Superconformal Algebras

In this section we study the structure of representations of conformal and superconformal algebras. Our goal is to understand which representations, or combinations of representations, are protected. This will allow us to show that all protected information that can be obtained by using group theory alone is captured by the Index we will define.

### 2.2.1 The 4 dimensional Conformal Algebra

The set of Killing vectors $M_{\mu\nu} = -i(x_\mu\partial_\nu - x_\nu\partial_\mu)$, $P_\mu = -i\partial_\mu$, $K_\mu = i(2x_\mu x.\partial - x^2\partial_\mu)$ and $H = x.\partial$ form a basis for infinitesimal conformal diffeomorphisms of $R^4$.

These Killing vectors generate the algebra

$$\begin{aligned}
[H, P_\mu] &= P_\mu, \\
[H, K_\mu] &= -K_\mu, \\
[K_\mu, P_\nu] &= 2(\delta_{\mu\nu} H - i M_{\mu\nu}), \\
[M_{\mu\nu}, P_\rho] &= i(\delta_{\mu\rho} P_\nu - \delta_{\nu\rho} P_\mu), \\
[M_{\mu\nu}, K_\rho] &= i(\delta_{\mu\rho} K_\nu - \delta_{\nu\rho} K_\mu), \\
[M_{\mu\nu}, M_{\rho\sigma}] &= i\left(\delta_{\mu\rho} M_{\nu\sigma} + \delta_{\nu\sigma} M_{\mu\rho} - \delta_{\mu\sigma} M_{\nu\rho} - \delta_{\nu\rho} M_{\mu\sigma}\right).
\end{aligned} \tag{2.2}$$

Consider a 4 dimensional Euclidean quantum field theory. It is sometimes possible to combine the conformal Killing symmetries of the previous paragraph with suitable action on fields to generate a symmetry of the theory. In such cases the theory in question is called a conformal field theory (CFT). The Euclidean path integral of a CFT may be given a useful Hilbert space interpretation via a radial quantization. Wave functions (kets) are identified with the path integral, with appropriate operator insertions, over the unit 3 ball surrounding the origin. Dual wave functions (bras) are obtained by acting on kets with by the conformal symmetry corresponding to inversions $x^\mu = \frac{x^\mu}{x^2}$ [1]. Under an inversion, the Killing vectors of the previous paragraph transform as $M_{\mu\nu} \to M_{\mu\nu}$, $H \to -H$, $P_\mu \to K_\mu$. As a consequence, the operators $M_{\mu\nu}, P_\mu, K_\nu$ are represented on the CFT Hilbert space (2.2) with the hermiticity conditions

$$M_{\mu\nu} = M_{\mu\nu}^\dagger, \quad P_\mu = K_\mu^\dagger. \tag{2.3}$$

Radial quantization of the CFT on $R^4$ is equivalent to studying the field theory on

[1]As a consequence, a bra may be thought of as being generated by a path integral, performed with appropriate insertions, on $R^4$ minus the unit 3 ball. The scalar product between a bra and a ket is the path integral - with insertions both inside and outside the unit sphere - over all of space.

$S^3\times$ time. The operators $M_{\mu\nu}$ generate the $SO(4)$ rotational symmetries of $S^3$, and $H$ is the Hamiltonian. From this point of view the conjugate generators $P_\mu$ and $K_\mu$ are less familiar; they act as ladder operators, respectively raising and lowering energy by a single unit.

The Hilbert space of a CFT on $S^3\times$ time may be decomposed into a sum of irreducible unitary representations of the conformal group. The theory of these representations was studied in detail by [5]. We present a brief review below, as a warm up for the superconformal algebra (see [6] and references therein for a recent discussion).

### 2.2.2 Unitary Representations of the Conformal Group

Any irreducible representation of the conformal group can be written as some direct sum of representations, $R^i_{compact}$, of the compact subgroup $SO(4)\times SO(2)$:

$$R_{SO(4,2)} = \sum_i \bigoplus R^i_{compact}. \tag{2.4}$$

The states within a given $SO(4)\times SO(2)$ representation all have the same energy. As the energy spectrum of any sensible quantum field theory is bounded from below, the representations of interest to us all possess a particular set of states with minimum energy. We will call these states (which we will take to transform as $R^\lambda_{compact}$) the lowest weight states. The $K^\mu$ operators necessarily annihilate all the states in $R^\lambda_{compact}$ because the $K^\mu$ have negative energy. We can now act on these lowest weight states with an arbitrary number of $P_\mu$ ('raising') operators to generate the remaining states in the representation. We will use the charges of the lowest weight state $|\lambda\rangle \equiv |E, j_1, j_2\rangle$ to label this representation. We use the fact that $SO(4) = SU(2)\times SU(2)$;

$j_1$ and $j_2$ are standard representation labels of these $SU(2)s$.

It is important that not all values of $E, j_1, j_2$ yield unitary representations of the conformal group. For a representation to be unitary, it is necessary for all states to have positive norm. Acting on the lowest energy states with $P_\mu$, we obtain (via a Clebsh Gordan decomposition) states that transform in the representations $(E+1, j_1 \pm 1/2, j_2 \pm 1/2)$. The norm of these states may be calculated using the commutation relations (2.2) [7] . The states with lowest norm turn out to have quantum numbers $(E+1, j_1 - \frac{1}{2}, j_2 - \frac{1}{2})$, and this norm is given by

$$\frac{\| \ \|^2}{2} = E - j_1 - 1 + \delta_{j_1 0} - j_2 - 1 + \delta_{j_2 0}. \tag{2.5}$$

Unitarity then requires that

$$\begin{aligned} &(i) \quad E \geq j_1 + j_2 + 2 \quad j_1 \neq 0 \quad j_2 \neq 0, \\ &(ii) \quad E \geq j_1 + j_2 + 1 \quad j_1 j_2 = 0. \end{aligned} \tag{2.6}$$

The special case $j_1 = j_2 = 0$ has an additional complication. In this case the norm of the level 2 state $P^2|\psi\rangle$ is proportional [7] to $E(E-1)$ and so is negative for $0 < E < 1$. The representation with $E = 0$ is annihilated by all momentum operators and represents the vacuum state. The representation at $E = 1$ is short and it obeys the equation $P^2|E\rangle = 0$ so it is a free field in the conformal field theory.

Unitary representations exist even when this bound is *strictly saturated*. The zero norm states, and all their descendants, are simply set to zero in these representations [2] making them shorter than generic.

[2]The consistency of this procedure relies on the fact that, at the unitarity bound, zero norm states are orthogonal to all states in the representation. As a consequence the inner product on the representation modded out by zero norm states is well defined and positive definite.

Now consider a one parameter (fixed line) of conformal field theories. An infinitesimal variation of the parameter that labels the theory will, generically, result in an infinitesimal variation in the energy of all the long representations of the theory. However a short representation can change its energy only if it turns into a long representation. In order for this to happen without a discontinuous jump in the spectrum of the CFT (i.e. a phase transition), it must pair up with some other representation, to make up the states of a long representation with energy at just above the unitarity threshold. Groups of short representations that can pair up in this manner are not protected; the numbers of such representations can jump discontinuously under infinitesimal variations of a theory.

However consider an Index $I$ that is defined as a sum of the form

$$I = \alpha[i]n[i] \tag{2.7}$$

where $i$ runs over the various short representations of the theory, $n[i]s$ are the number of representations of the $i^{th}$ variety, and $\alpha[i]$ are fixed numbers chosen so that $I$ evaluates to zero on any collection of short representations that can pair up into long representations. By definition such an Index is unaffected by groups of short representations pairing up and leaving, as it evaluates to zero anyway on any set of representations that can. It follows that an Index cannot change under continuous deformations of the theory.

We will now argue that the conformal algebra does not admit any non trivial Indices. In order to do this we first list how a long representation decomposes into a sum of other representations (at least one of which is short) when its energy is decreased so that it hits the unitarity bound. Let us denote the representations as follows. $A_{E,j_1,j_2}$

denotes the generic long representation, $C_{j_1,j_2}$ denotes the short representations with $j_1$ and $j_2$ both not equal to zero, $B^L_{j_1}$ denotes the short representations with $j_2 = 0$, $B^R_{j_2}$ the short ones with $j_1 = 0$. Finally we denote the special short representation at $E = 1$ and $j_1 = j_2 = 0$ by $B$. As the energy is decreased to approach the unitarity bound we find

$$
\begin{aligned}
&\lim_{\epsilon\to 0} \chi[A_{j_1+j_2+2+\epsilon,j_1,j_2}] = \chi[C_{j_1,j_2}] + \chi[A_{j_1+j_2+3,j_1-\frac{1}{2},j_2-\frac{1}{2}}] \\
&\lim_{\epsilon\to 0} \chi[A_{j_1+1+\epsilon,j_1,0}] = \chi[B^L_{j_1}] + \chi[C_{j_1-\frac{1}{2},\frac{1}{2}}] \\
&\lim_{\epsilon\to 0} \chi[A_{j_2+1+\epsilon,0,j_2}] = \chi[B^R_{j_2}] + \chi[C_{\frac{1}{2},j_2-\frac{1}{2}}] \\
&\lim_{\epsilon\to 0} \chi[A_{1+\epsilon,0,0}] = \chi[B] + \chi[A_{3,0,0}].
\end{aligned}
\tag{2.8}
$$

In (2.8) and throughout this paper, the symbol $\chi$ denotes the super-character on a representation[3].

It follows from (2.8) that $\sum_i \alpha_i n_i$ is an Index only if

$$
\alpha_{C_{j_1,j_2}} = 0, \quad \alpha_{B^L_{j_1}} + \alpha_{C_{j_1-\frac{1}{2},\frac{1}{2}}} = 0, \quad \alpha_{B^R_{j_2}} + \alpha_{C_{\frac{1}{2},j_2-\frac{1}{2}}} = 0, \quad \alpha_B = 0. \tag{2.9}
$$

The only solution to these equations has all $\alpha$ to zero; consequently the conformal algebra admits no nontrivial Indices. The superconformal algebra will turn out to be more interesting in this respect.

[3]i.e. $Tr_R(-1)^F G$ where $R$ is an arbitrary representation, $G$ is an arbitrary group element, and $F$ is the Fermion number, which plays no role in the representation theory of the conformal group, but will be important when we turn superconformal groups below.

### 2.2.3 Unitary Representations of $d = 4$ Superconformal Algebras

In the next two subsections we review the unitary representations of the $d = 4$ superconformal algebras [8] that were studied in [9, 10, 11, 7, 12, 13]. A supersymmetric field theory that is also conformally invariant, actually enjoys superconformal symmetry, a symmetry algebra that is larger than the union of conformal and super symmetry algebras. The bosonic subalgebra of the $\mathcal{N} = m$ superconformal algebra consists of the conformal algebra times $U(m)$, except in the special case $m = 4$, where the $R$ symmetry algebra is $SU(4)$. The fermionic generators of this algebra consist of the $4m$ supersymmetry generators $Q^{\alpha i}$ and $\bar{Q}_i^{\dot{\alpha}}$, together with the super conformal generators $S_{\alpha i}, \bar{S}_{\dot{\alpha}}^j$. The generators transform under $SO(4) \times U(m)$ as indicated by their Index structure (an upper $i$ Index indicates a $U(m)$ fundamental, while a lower $i$ Index is a $U(m)$ anti-fundamental). The commutation relations of the algebra may be found, for example, in Appendix A.1 of [14]. Here, we will only need,

$$\{S_{\alpha i}, Q^{\beta j}\} = \delta_i^j (J_1)_\alpha^\beta + \delta_\alpha^\beta R_i^j + \delta_i^j \delta_\alpha^\beta (\frac{H}{2} + r\frac{4-m}{4m}) \tag{2.10}$$

where $(J_1)_\alpha^\beta$ are the $SU(2)$ generators in spinor notation, $R_i^j$ are the $SU(m)$ generators and $r$ is the $U(1)$ generator. As in the previous subsection, radial quantization endows these generators with hermiticity properties; specifically

$$(Q^{\alpha i})^\dagger = S_{\alpha i}, \quad (\bar{Q}_i^{\dot{\alpha}})^\dagger = \bar{S}_{\dot{\alpha}}^i \tag{2.11}$$

The theory of unitary representations of the superconformal algebra is similar to that of the conformal algebra. Irreducible representations are labeled by the energy $E$ and the $SU(2) \times SU(2)$ and $U(m)$ representations of their lowest weight states.

We label $U(m)$ representations by their $U(1)$ charge $r$ (normalized such that $Q^{\alpha i}$ has charge +1 and $\bar{Q}_i^{\dot{\alpha}}$ has charge -1) and the integers $R_k$ ($k = 1 \dots m-1$), the number of columns of height $k$ in the Young Tableaux for the representation.[4]

Lowest weight states are annihilated by the $S$ but, generically, not by the $Q$ operators. $Q^{\alpha i}$ have $E = \frac{1}{2}$ and transform in the $SU(2)\times SU(2)\times U(m)$ representation with quantum numbers $j_1 = \frac{1}{2}, \quad j_2 = 0, \quad r = 1, \quad R_1 = 1, \quad R_i = 0 \quad (i > 1)$. Let $|\psi_a\rangle$ be the set of lowest weight states of this algebra that transforms in the representation $(E, j_1, j_2, r, R_i)$. The states $Q^{\alpha i}|\psi_a\rangle$ transform in all the Clebsh Gordan product representations; the lowest norm among these states occurs for those that have quantum numbers $(E + \frac{1}{2}, j_1 - \frac{1}{2}, j_2, r + 1, R_1 + 1, R_j)$; the value of the norm of these states is given by [7]

$$\begin{aligned} 2\|\chi_1\|^2 &= E + 2\delta_{j_1,0} - E_1(j_1, r, R_i) \ , \\ E_1 &\equiv 2 + 2j_1 + 2\frac{\sum_{p=1}^{m-1}(m-p)R_p}{m} + \frac{r(4-m)}{2m}. \end{aligned} \tag{2.12}$$

In a similar fashion, of the states of the form $\bar{Q}_{\dot{\alpha} i}|\psi\rangle$ the lowest norm occurs for those that transform in $(E + \frac{1}{2}, j_1, j_2 - \frac{1}{2}, r - 1, R_k, R_{m-1} + 1)$, and the norm of these states is equal to [7]

$$\begin{aligned} 2\|\chi_2\|^2 &= E + 2\delta_{j_2,0} - E_2(j_1, j_2, r, R_i) \\ E_2 &\equiv 2 + 2j_2 + \frac{2\sum_{p=1}^{m-1} pR_p}{m} - \frac{r(4-m)}{2m}. \end{aligned} \tag{2.13}$$

Clearly unitarity demands that $\|\chi_1\|^2 \geq 0$ and $\|\chi_2\|^2 \geq 0$. As for the conformal group, representations with either $\|\chi_1\|^2 = 0$ or $\|\chi_2\|^2 = 0$ or both zero are allowed.

[4] $R_k$ may also be thought of as the eigenvalues of the highest weight vectors under the diagonal generator $R_k$ whose $k^{th}$ diagonal entry is unity, $(k+1)^{th}$ entry is $-1$, and all other are zero, in the defining representation.

In such representations the zero norm states and all their descendants are simply set to zero, yielding short representations.

In the special case $j_1 = 0$ the positivity of the norm at level 2 yields more information. Of states of the form $Q^{\alpha i}Q^{\beta j}|\psi_a\rangle$ (where $|\psi_a\rangle$ are the lowest weight states), those that have the smallest norm transform in the representation $(E+1, 0, j_2, r+2, R_1+2, R_j)$. The norm of these states turns out to be proportional to $(E-E_1)(E-E_1+2)$ where $E_1$ is defined in (2.12). Thus unitarity disallows representations in the window $E_1 - 2 < E < E_1$. Representations at $E = E_1 - 2$ and $E = E_1$ are both short and both allowed. Representations at $E = E_1 - 2$ are special because they are separated from long representations (with the same value of all other charges) by an energy gap of two units. All these remarks also apply to the special case $j_2 = 0$, upon replacing $Q^{\alpha i}$ with $\bar{Q}_i^{\dot{\alpha}}$ and $E_1$ with $E_2$.

In [12], Dolan and Osborn, performed a comprehensive tabulation of short representations of the $d = 4$ superconformal algebras. We will adopt a notation that is slightly different from theirs. Representations are denoted by $\mathbf{x^L x^R}_{E,j_1,j_2,r,R_i}$ where

$$\mathbf{x^L} = \begin{cases} \mathbf{a} & \text{if } E > E_1 \\ \mathbf{c} & \text{if } E = E_1 \text{ and } j_1 \geq 0 \\ \mathbf{b} & \text{if } E = E_1 - 2 \text{ and } j_1 = 0 \end{cases} \tag{2.14}$$

and

$$\mathbf{x^R} = \begin{cases} \mathbf{a} & \text{if } E > E_2 \\ \mathbf{c} & \text{if } E = E_2 \text{ and } j_2 \geq 0 \\ \mathbf{b} & \text{if } E = E_2 - 2 \text{ and } j_2 = 0 \end{cases} \tag{2.15}$$

Further, we will usually omit to specify the first (energy) subscript on all short representations as this energy is determined by the other charges. Thus representations denoted by **aa** are long; all other representations are short.

### 2.2.4 The Null Vectors in Short Representations

We now study the nature of the null vectors in short representations in more detail. Consider a representation of the type **cx**, with $j_1 > 0$, where **x** is either $\mathbf{a}, \mathbf{c}$ or **b**. Such a representation has $\|\chi_1\|^2 = 0$. The descendants of the null-state form another representation of the superconformal algebra. This representation also has null states[5] characterized by their own value of $(\|\chi_1'\|^2, \|\chi_2'\|^2)$. A short calculation [6]shows that $\|\chi_1'\|^2, \|\chi_2'\|^2)/ = (0, \|\chi_2\|^2)$ . It follows that the $Q$ null states of a representation of type **cx** are generically also of the type **cx**. The exception to this rule occurs when $j_1 = 0$, in which case the null states are of type **bx**. Of course analogous statements are also true for $\bar{Q}$ null states. All of this may be summarized in a set of decomposition formulae, for the supercharacters,

$$\chi[R] = \mathrm{Tr}_R\left[(-1)^{2(J_1+J_2)}G\right], \tag{2.16}$$

where $G$ is an arbitrary element of the superconformal group. These formulae describe how a long representation decomposes into a set of short representation when its

[5]When we say that a short representation has 'null states' of a particular type we mean the following. When we lower the energy of a long representation down to its unitarity bound ($E_1$ or $E_2$), the long representation splits into a positive norm short representation $m$, plus a set of null representations $m'$. We describe this situation by the words 'the short representation $m$ has null representations $m'$. As is clear from this definition, it is meaningless to talk of the null state content of representations of the sort **bx** or **xb**, as these representations are separated by a gap from long representations.

[6]$(\|\chi_1'\|^2, \|\chi_2'\|^2) = (\frac{1}{2}+1-2(m-1)/m-(4-m)/2m, \|\chi_2\|^2-\frac{1}{2}+2/m+(4-m)/2m) = (0, \|\chi_2\|^2)$.

energy hits the unitarity bound.

$$\begin{aligned}
&\lim_{\epsilon\to 0}\chi[\mathbf{aa}_{E_1+\epsilon,j_1,j_2,r,R_i}] = \chi[\tilde{\mathbf{c}}\mathbf{a}_{j_1,j_2,r,R_i}] + \chi[\tilde{\mathbf{c}}\mathbf{a}_{j_1-\frac{1}{2},j_2,r+1,R_1+1,R_j}],\ E_1 > E_2 \\
&\lim_{\epsilon\to 0}\chi[\mathbf{aa}_{E_2+\epsilon,j_1,j_2,r,R_i}] = \chi[\mathbf{a}\tilde{\mathbf{c}}_{j_1,j_2,r,R_i}] + \chi[\mathbf{a}\tilde{\mathbf{c}}_{j_1,j_2-\frac{1}{2},r-1,R_k,R_{m-1}+1}],\ E_2 > E_1 \\
&\lim_{\epsilon\to 0}\chi[\mathbf{aa}_{E_2+\epsilon,j_1,j_2,r,R_i}] = \chi[\tilde{\mathbf{c}}\tilde{\mathbf{c}}_{j_1,j_2,r,R_i}] + \chi[\tilde{\mathbf{c}}\tilde{\mathbf{c}}_{j_1-\frac{1}{2},j_2,r+1,R_1+1,R_j}]+ \\
&\chi[\tilde{\mathbf{c}}\tilde{\mathbf{c}}_{j_1,j_2-\frac{1}{2},r-1,R_k,R_{m-1}+1}] + \chi[\tilde{\mathbf{c}}\tilde{\mathbf{c}}_{j_1-\frac{1}{2},j_2-\frac{1}{2},r,R_1+1,R_l,R_{m-1}+1}],\ E_1 = E_2
\end{aligned} \tag{2.17}$$

where, in this equation and, as far as possible, in the rest of the paper, we use the Index convention

$$i = 1\ldots m-1,\quad j = 2\ldots m-1,\quad k = 1\ldots m-2,\quad l = 2\ldots m-2. \tag{2.18}$$

On the right hand side of (2.17) we have used the notation given in table 2.2.4.

Table 2.1: Short Representations

| Symbol | Denotation |
|---|---|
| $\tilde{\mathbf{c}}\mathbf{a}_{j_1,j_2,r,R_i}$ | $\mathbf{ca}_{j_1,j_2,r,R_i},\quad j_1 \geq 0$<br>$\mathbf{ba}_{0,j_2,r+1,R_1+1,R_j},\quad j_1 = -\frac{1}{2}$ |
| $\mathbf{a}\tilde{\mathbf{c}}_{j_1,j_2,r,R_i}$ | $\mathbf{ac}_{j_1,j_2,r,R_i},\quad j_2 \geq 0$<br>$\mathbf{ab}_{j_1,0,r-1,R_k,R_{m-1}+1},\quad j_2 = -\frac{1}{2}$ |
| $\tilde{\mathbf{c}}\tilde{\mathbf{c}}_{j_1,j_2,r,R_i}$ | $\mathbf{cc}_{j_1,j_2,r,R_i},\quad j_1 \geq 0, j_2 \geq 0$<br>$\mathbf{cb}_{j_1,0,r-1,R_k,R_{m-1}+1},\quad j_2 = \frac{-1}{2}, j_1 \geq 0$<br>$\mathbf{bc}_{0,j_2,r+1,R_1+1,R_j},\quad j_1 = -\frac{1}{2}, j_2 \geq 0$<br>$\mathbf{bb}_{0,0,r,R_1+1,R_l,R_{m-1}+1},\quad j_1 = j_2 = -\frac{1}{2}$ |

### 2.2.5 Indices For Four Dimensional Super Conformal Algebras

We now turn to a study of the Indices for these algebras. Using (2.17) it is not difficult to convince oneself that the set of Indices for the four dimensional superconformal field theories is a vector space that is spanned by

1. The number of representations of the sort **bx** with $R_1 = 0$ or $R_1 = 1$ plus the number of representations of the sort **xb** with $R_{m-1} = 0$ or $R_{m-1} = 1$.

2. The Indices
$$I^L_{j_2,\hat{r},M,R_j} = \sum_{p=-1}^{M} (-1)^{p+1} n[\tilde{\mathbf{c}}\mathbf{x}_{\frac{p}{2},j_2,\hat{r}-p,M-p,R_j}] \tag{2.19}$$
for all values of $\hat{r}$ and non negative integral values of $j_2, M, R_j$. In the case $m = 1$ we do not have the Indices $M$ or $R_j$ and the sum runs from $p = -1$ to infinity. In the $m = 4$ case, simply ignore the $r$ and $\hat{r}$ subIndices.

3. The Indices
$$I^R_{j_1,r'',R_k,N} = \sum_{p=-1}^{M} (-1)^{p+1} n[\mathbf{x}\tilde{\mathbf{c}}_{j_1,\frac{p}{2},r''+p,R_k,N-p}] \tag{2.20}$$
for all values of $r''$ and non negative integral values of of $j_1, R_k, N$, with the same remarks for $m = 1, 4$.

In the special case that representations that contribute to the sum in (2.19) and (2.20) have quantum numbers on which $E_1 = E_2$[7], the Indices (2.19) and (2.20) are subject to the additional constraints

$$\sum_{p=-1}^{N} (-1)^p I^L_{\frac{p}{2},r'''+p,M,R_l,N-p} = \sum_{p=-1}^{M} (-1)^p I^R_{\frac{p}{2},r'''-p,M-p,R_l,N} \quad (E_1 = E_2) \tag{2.21}$$

for all values of $r''' = -\infty \ldots \infty$, and non negative integral values of $M, N, R_l$.

These results are explained in more detail in Appendix B.1 of [14] which also presents a detailed listing of the absolutely protected multiplets, for the $\mathcal{N} = 1, 2, 4$ superconformal algebras.

[7]If this relation is true for any term that contributes to the sum, it is automatically true on all other terms as well.

## 2.3 A Trace Formula for the Index

The supercharges $Q^{\alpha i}$ transform in the fundamental or $(1,0,\ldots,0)$ representation of $SU(m)$. Let $Q \equiv Q^{-\frac{1}{2},1}$, the supercharge whose $SU(2)\times SU(2)$ quantum numbers are $(j_1^3, j_2^3) = (-\frac{1}{2}, 0)$, that has $r=1$, and whose $SU(m)$ quantum numbers are $(1,0,\ldots 0)$. Let $S \equiv Q^\dagger$. Then (see (2.10))

$$2\{S,Q\} = H - 2J_1 - 2\sum_{k=1}^{m-1}\frac{m-k}{m}R_k - \frac{(4-m)r}{2m} = E - (E_1 - 2) \equiv \Delta. \tag{2.22}$$

It follows from (2.22) that every state in a unitary representation of the superconformal group has $\Delta \geq 0$. Note that the Jacobi identity implies that $Q$ and $S$ commute with $\Delta$.

Consider a unitary representation $R$ of the superconformal group that is not necessarily irreducible. Let $R_{\Delta_0}$ denote the linear vector space of states with $\Delta = \Delta_0 > 0$. It follows from (2.22) that if $|\psi\rangle$ is in $R_{\Delta_0}$ then

$$|\psi\rangle = Q\frac{S}{\Delta_0}|\psi\rangle + S\frac{Q}{\Delta_0}|\psi\rangle. \tag{2.23}$$

Let $R^Q_{\Delta_0}$ denote the subspace of $R_{\Delta_0}$ consisting of states annihilated by $Q$ and $R^S_{\Delta_0}$ the set of states in $R_{\Delta_0}$ that are annihilated by $S$. It follows immediately from (2.23) (and the unitarity of the representation) that $R_{\Delta_0} = R^Q_{\Delta_0} + R^S_{\Delta_0}$ and that $S|\psi\rangle = |\psi'\rangle$ is a one to one map from $R^Q_{\Delta_0}$ to $R^S_{\Delta_0}$ ($Q/\Delta_0$ provides the inverse map).

Now consider the Witten Index

$$\mathcal{I}^{WL} = Tr_R\left[(-1)^F \exp(-\beta\Delta + M)\right] \tag{2.24}$$

where $M$ is any element of the subalgebra of the superconformal algebra that commutes with $Q$ and $S$. We discuss this subalgebra in detail in the next subsection.

It follows that the states in $R_{\Delta_0}$ do not contribute to this Index, the contribution of $R^Q_{\Delta_0}$ cancels against that of $R^S_{\Delta_0}$. Consequently, $\mathcal{I}^{WL}$ receives contributions only from states with $\Delta = 0$, i.e. those states that are annihilated by both $Q$ and $S$. Thus, despite appearances, (2.24) is independent of $\beta$. As no long representation contains states with $\Delta = 0$, such representations do not contribute to $\mathcal{I}^{WL}$. It also follows from continuity that $\mathcal{I}^{WL}$ evaluates to zero on groups of short representations that a long representation breaks up into when it hits unitarity threshold. As a consequence $\mathcal{I}^{WL}$ is an Index; it cannot change under continuous variations of the superconformal theory, and must depend linearly on the Indices, $I^L$ and $I^R$, listed in the previous section. We will explain the relationship between $\mathcal{I}^{WL}$ and $I^L$ in more detail in subsection 3.2 and 3.3 below. The main result of the following subsections is to show that (2.24) (and its $\mathcal{I}^{WR}$ version) completely capture the information contained in the Indices defined in the previous section, which is all the information about protected representations that can be obtained without invoking any dynamical assumption.

### 2.3.1 The Commuting Subalgebra

In this subsection we briefly describe the subalgebra of the superconformal algebra that commutes with the $SU(1|1)$ algebra spanned by $Q, S, \Delta$.

The $\mathcal{N} = m, d = 4$ superconformal algebra is the super-matrix algebra $SU(2,2|m)$.[8] Supersymmetry generators transform as bifundamentals under the bosonic subgroups $SU(2,2)$ and $SU(m)$. It is not difficult to convince oneself that the commuting sub-

[8]For $m = 4$ we have $PSU(2,2|4)$.

algebra we are interested in is $SU(2,1|m-1)$[9]. The generators of $SU(2,1|m-1)$ are related to those of $SU(2,2|m)$ via the obvious reduction. In more detail, the bosonic subgroup of $SU(2,1|m-1)$ is $SU(2,1)\times U(m-1)$. The $U(m-1)$ factor sits inside the superconformal $U(m)$ setting all elements the first row and first column to zero, except for the 11 element which is set to one. The Cartan elements $(E', j_2', r', R_i')$ of the subalgebra are given in terms of those for the full algebra by

$$E' = E + j_1, \quad j_2' = j_2, \quad r' = \frac{(m-1)r}{m} - \sum_{p=1}^{m-1} \frac{m-p}{m} R_p, \quad R_k' = R_{k+1}. \tag{2.25}$$

where $R_k'$ are the Cartan elements of $U(m-1)$ and $r'$ is the $U(1)$ charge in $U(m-1)$. We will think of (2.25) as defining a (many to one) map from $(E, j_1, j_2, r, R_i)$ to $(E', j_2', r', R_i')$

We will be interested in the representations of the subalgebra, $SU(2,1|m-1)$, that are obtained by restricting a representation of the full algebra, $SU(2,2|m)$, to states with $\Delta = 0$. Null vectors, if any, of $SU(2,1|m-1)$ are inherited from those of $SU(2,2|m)$. It is possible to show that $SU(2,1|m-1)$ is short only when $SU(2,2|m)$ is one of the representations $cb, cc$ or if $R$ is $bx$ with $R_1 = 0$. We direct the reader to Appendix B.3 of [14] for a further discussion of this issue.

### 2.3.2 $\mathcal{I}^{WL}$ expanded in sub-algebra characters with $I^L$ as coefficients

In this subsection we present a formula for Index $\mathcal{I}^{WL}$ as a sum over super characters of the commuting subalgebra, $SU(2,1|m-1)$.

[9] Or $PSU(2,1|3)$ for $m = 4$.

It is not difficult to convince oneself that on any short irreducible representation $R$ of the superconformal algebra $SU(2,2|m)$, $\mathcal{I}^{WL}$ evaluates to the supercharacter of a single irreducible representation $R'$ of the subalgebra $SU(2,1|m-1)$. More specifically we find

$$\begin{aligned}\mathcal{I}^{WL}[\mathbf{bx}_{0,j_2,r,R_i}] =&\chi_{sub}[\vec{b}]\\ \mathcal{I}^{WL}[\mathbf{cx}_{j_1,j_2,r,R_i}] =&(-1)^{2j_1+1}\chi_{sub}[\vec{c}]\end{aligned} \tag{2.26}$$

where $\chi_{sub}$ is the supercharacter

$$\chi_s[R'] = \mathrm{Tr}_{R'}\left[(-1)^{2J_2}G'\right], \tag{2.27}$$

where $G'$ is an element of the Cartan subgroup. The vectors $\vec{b}$ and $\vec{c}$ specify the highest weight of the representation of the subalgebra in the Cartan basis $[E', j_2', r', R_k']$ defined in (2.25).

$$\begin{aligned}\vec{b} &= [\frac{3}{2}r - 2r', j_2, r', R_j],\\ \vec{c} &= [3 + 3(j_1 + r/2) - 2r', j_2, r', R_j]\end{aligned} \tag{2.28}$$

where $r'$ is the function defined in (2.25); we emphasize the fact that it depends on $r$ and $R_1$ only through the combination $r - R_1$.

Notice that the functions that specify the character of the subalgebra, (2.28), are not one to one. In fact, it follows from (2.26), (2.28), that $\mathcal{I}^{WL}$ evaluates to the same subalgebra character for each representation $R$ that appears in the sum in (2.19), for fixed values of $j_2, r', M, R_i$. Notice that by formally setting $j_1 = -1/2$ in the second line of (2.28) we get the Cartan values for the subalgebra that we expect for the representation $\mathbf{b}$ according to the definition of $\tilde{\mathbf{c}}$ in table 2.2.4. This implies that we can replace $\mathbf{c}$ in (2.26) by $\tilde{\mathbf{c}}$. More specifically

$$\mathcal{I}^{WL}[\tilde{\mathbf{c}}_{\frac{p}{2},j_2,\hat{r}-p,M-p,R_j}] = (-1)^p \mathcal{I}^{WL}[\tilde{\mathbf{c}}_{0,j_2,\hat{r},M,R_j}] \tag{2.29}$$

It follows immediately from (2.26), (2.29), that $\mathcal{I}^{WL}$, evaluated on any (in general reducible representation) $A$ of the superconformal algebra evaluates to

$$\begin{aligned}\mathcal{I}^{WL}[A] = &\sum_{j_2,r,R_i} \Big(n[\mathbf{bx}_{0,j_2,r,0,R_i}]\chi_{sub}[\vec{b}_0] + n[\mathbf{bx}_{0,j_2,r,1,R_i}]\chi_{sub}[\vec{b}_1]\Big) \\ &+ \sum_{j_2,r',M,R_i} I^L_{j_2,\hat{r},M,R_i}\chi_{sub}[\vec{c}_0]\end{aligned} \tag{2.30}$$

Where $\vec{b}_{0,1}$ are given by (2.28) with $R_1 = 0, 1$ respectively and $\vec{c}_0$ is given by (2.28) with $j_1 = 0$, $r = \hat{r}$, $R_1 = M$. The quantities $n[\mathbf{xx}_{j_1,j_2,r,R_i}]$ in (2.30) are the number of copies of the irreducible representation, with listed quantum numbers, that appear in $A$, and $I^L_{j_2,\hat{r},M,R_i}$ are the Indices (2.19) made out of these numbers.

Notice that most of the discussion in this section goes through unchanged if we were to consider the supergroup $SU(2|4)$ (or $SU(2|m)$). The representation theory of this group was studied in [15, 16] and the Index was used in [17] to analyze various field theories with this symmetry. The Index for the plane wave matrix model is given by an expression like (4.3) below but without the denominators (this is then inserted into (4.1)). Notice that the fact that the Index for $N = 4$ Yang Mills and the Index for the plane wave matrix model are different implies that we cannot continuously interpolate between $\mathcal{N} = 4$ super Yang Mills and the plane wave matrix model while preserving the $SU(2|4)$ symmetry. In [18] BPS representations and an Index for $SU(1|4)$ were considered.

### 2.3.3 The Witten Index $\mathcal{I}^{WR}$

As in Section 2, we may define a second Index $\mathcal{I}^{WR}$. The theory for this Index is almost identical. We focus on the supercharge, $\bar{Q}^{-\frac{1}{2}}_{m-1}$ which has $SU(2)\times SU(2)$ quantum numbers, $(j_1^3, j_2^3) = (0, -\frac{1}{2})$, $r = -1$ and $SU(4)$ quantum numbers $(0, 0, \ldots, 1)$. Let $\bar{S} = \bar{Q}^\dagger$.

It is then easy to verify that

$$2\{\bar{S}, \bar{Q}\} = H - 2J_2 - 2\sum_{k=1}^{m-1} \frac{k}{m} R_k + \frac{(4-m)r}{2m} = E - (E_2 - 2) \equiv \bar{\Delta}. \tag{2.31}$$

It follows from (2.31) that every state in a unitary representation of the superconformal group has $\bar{\Delta} \geq 0$.

Following (2.24) we define

$$\mathcal{I}^{WR} = Tr_R\left[(-1)^F \exp(-\beta\bar{\Delta} + \bar{M})\right], \tag{2.32}$$

where $\bar{M}$ is the part of the superconformal algebra that commutes with $\bar{Q}$ and $\bar{S}$.

The Cartan elements of this subalgebra are given in terms of those of the algebra by

$$E' = E + j_2, \quad j_1' = j_1, \quad r' = \frac{(m-1)(r + R_{m-1})}{m} + \sum_{p=1}^{m-2} \frac{p}{m} R_p, \quad R_k' = R_k. \tag{2.33}$$

Note that $r'$ depends on $r$ and $R_{m-1}$ only through the combination $r + R_{m-1}$. We then find that the Index (2.32) is zero on long representations and for $\mathbf{c}, \mathbf{b}$ representations it is equal to

$$\begin{aligned} \mathcal{I}^{WR}[\mathbf{bx}_{j_1,0,r,R_i}] &= \chi_{sub}[\vec{\bar{b}}] \\ \mathcal{I}^{WR}[\mathbf{cx}_{j_1,j_2,r,R_i}] &= (-1)^{2j_2+1}\chi_{sub}[\vec{\bar{c}}] \end{aligned} \tag{2.34}$$

where the representation of the subalgebra is specified by the vectors $\vec{\bar{b}}$, $\vec{\bar{c}}$ in the basis $[E', j_1', r', R_k']$ specified by (2.33).

$$\begin{aligned} \vec{\bar{b}} &= [-\frac{3}{2}r + 2r', j_1, r'(r + R_{m-1}, R_k), R_k], \\ \vec{\bar{c}} &= [3 + 3(j_2 - r/2) + 2r', j_1, r'(r + R_{m-1}, R_k), R_k]. \end{aligned} \tag{2.35}$$

where $r'$ is the function in (2.33) . We find that on a general representation (not necessarily irreducible) of the superconformal algebra, $\mathcal{I}^{WR}$ evaluates to

$$\begin{aligned}\mathcal{I}^{WR}[R] = &\sum_{j_1,r,R_i} \left( n[\mathbf{xb}_{j_1,0,r,R_k,0}]\chi_{sub}[\vec{\bar{b}}_0] + n[\mathbf{xb}_{j_1,0,r,R_k,1}]\chi_{sub}[\vec{\bar{b}}_1] \right) \\ &+ \sum_{j_1,r'',R_k,N} I^R_{j_1,r'',R_k,N}\chi_{sub}[\vec{\bar{c}}_0]\end{aligned} \tag{2.36}$$

Where $\vec{\bar{b}}_{0,1}$ are given by (2.35) with $R_{m-1} = 0, 1$ respectively and $\vec{\bar{c}}_0$ is given by (2.35) with $j_2 = 0$, $r = r''$, $R_{m-1} = N$. The quantities $n[\mathbf{xx}_{j_1,j_2,r,R_i}]$ in (2.30) are the number of copies of the irreducible representation, with listed quantum numbers, that appear in $R$, and $I^R_{j_1,r'',R_k,N}$ are the Indices (2.20) made out of these numbers.

The main lesson we should extract from (2.30), (2.36) is that each of the Indices defined in section two are multiplied by *different* $SU(1,2|m-1)$ (or $SU(2,1|m-1)$) characters in [2.30,2.36]. This shows that the Witten Indices (2.24) (2.32) capture all the protected that follows from the supersymmetry algebra alone.

## 2.4 Computation of the Index in $\mathcal{N} = 4$ Yang Mills on $S^3$

### 2.4.1 Weak Coupling

We will now evaluate the Index (2.24) for free $\mathcal{N} = 4$ Yang Mills on $S^3$. In the free theory this Index may be evaluated either by simply counting all gauge invariant states with $\Delta = 0$ and specified values for other charges [3, 4] or by evaluating a path integral [4]. The two methods give the same answer. We will give a very brief description of the path integral method, referring the reader to [4] for all details. One evaluates the path integral over the $\Delta = 0$ modes of all the fields of the $\mathcal{N} = 4$ theory on $S^3 \times S^1$ with periodic boundary conditions for the fermions around $S^1$ (to deal with the $(-1)^F$ insertion) and twisted boundary conditions on all charged fields (to insert the appropriate chemical potentials). While the path integral over all other modes may be evaluated in the one loop approximation, the path integral over the zero mode of $A_0$ on this manifold must be dealt with exactly (as the integrand lacks a quadratic term for this mode, the integral over it is always strongly coupled at every nonzero coupling no matter how weak). Upon carefully setting up the problem one finds that the integral over $A_0$ is really an integral over the holonomy matrix $U$, and the Index $\mathcal{I}^{WL}$ evaluates to

$$\mathcal{I}_{YM} = \int [dU] \exp\left\{\sum \frac{1}{m} f(t^m, y^m, u^m, w^m) \mathrm{tr}(U^\dagger)^m \mathrm{tr} U^m\right\} \tag{2.37}$$

where $f(t, y, u, w)$ is the Index $\mathcal{I}^{WL}$ evaluated on space of 'letters' or 'gluons' of the $\mathcal{N} = 4$ Yang Mills theory. As a consequence, in order to complete our evaluation of the Index (2.37) we must merely evaluate the single letter partition function $f$.

$f$ may be evaluated in many ways. Group theoretically, we note that the letters of Yang Mills theory transform in the 'fundamental' representation of the superconformal group (the representation whose quantum lowest weight state has quantum numbers $E = 1$, $j_1 = j_2 = 0$, $R_1 = R_3 = 0$ and $R_2 = 1$). $f$ is simply the supertrace over this representation and may be evaluated by purely group theoretic techniques.

It is useful, however, to re-derive this result in a more physical manner. The full set of single particle $\Delta = 0$ operators in Yang Mills theory is given by the fields listed in Table 2.2. below, acted on by an arbitrary numbers of the two derivatives $\partial_{+\pm}$ (see the last row of Table 2.2) modulo the single equation of motion listed in the second

Table 2.2: Letters with $\mathbf{\Delta = 0}$

| **Letter** | $(-\mathbf{1})^{\mathbf{F}}[\mathbf{E}; \mathbf{j_1}, \mathbf{j_2}]$ | $[\mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}]$ | $[\mathbf{R_1}, \mathbf{R_2}, \mathbf{R_3}]$ |
|---|---|---|---|
| $X, Y, Z$ | [1,0,0] | [1,0,0]+cyclic | [0,1,0]+[1,-1,1]+[1,0,-1] |
| $\psi_{+,0;-++}$ + cyc<br>$\psi_{0,\pm,+++}$ | $-[\frac{3}{2}, \frac{1}{2}, 0]$<br>$-[\frac{3}{2}, 0, \pm\frac{1}{2}]$ | $[-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$ + cyc<br>$[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$ | $[1, -1, 0], [0, 1, -1], [0, 0, 1]$<br>$[1, 0, 0]$ |
| $F_{++}$ | $[2, 1, 0]$ | $[0, 0, 0]$ | $[0, 0, 0]$ |
| $\sum_{\pm} \partial_{+\pm}\psi_{0,\mp;+++} = 0$ | $[\frac{5}{2}, \frac{1}{2}, 0]$ | $[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$ | $[1, 0, 0]$ |
| $\partial_{+\pm}$ | $[1, \frac{1}{2}, \pm\frac{1}{2}]$ | $[0, 0, 0]$ | $[0, 0, 0]$ |

last row of Table 2.2).

In Table 2.2 we have listed both the $SU(4)$ Cartan charges $R_1, R_2, R_3$ used earlier in this paper, as well as the $SO(6)$ Cartan charges, $q_1, q_2, q_3$ (the eigenvalues in each of the 3 planes of the embedding $R^6$) of all fields.

To find $f$ we evaluate (2.24) by summing over the letters

$$\begin{aligned} f &= \sum_{\text{letters}} (-1)^F t^{2(E+j_1)} y^{2j_2} v^{R_2} w^{R_3} \\ &= \frac{t^2(v + \frac{1}{w} + \frac{w}{v}) - t^3(y + \frac{1}{y}) - t^4(w + \frac{1}{v} + \frac{v}{w}) + 2t^6}{(1 - t^3 y)(1 - \frac{t^3}{y})}. \end{aligned} \tag{2.38}$$

Remarkably the expression for $1 - f$ factorizes

$$1 - f = \frac{(1 - t^2/w)(1 - t^2 w/v)(1 - t^2 v)}{(1 - t^3 y)(1 - t^3/y)} \tag{2.39}$$

The expression for $\mathcal{I}_{YM}^{WL}$ is well defined (convergent) only if $t, y, v, w$ have values such that every contributing letter has a weight of modulus $< 1$; applying this criterion to the three scalars and the two retained derivatives yields the restriction $t^2 v < 1, \quad t^2/w < 1, \quad t^2 v/w < 1, \quad t^3 y < 1, t^3/y > 1$. It follows from (2.39) that $f < 1$ for all legal values of chemical potentials.

We will now proceed to evaluate the integral in (2.37), using saddle point techniques, in the large $N$ limit (note, however, that (2.37) is the *exact* formula valid for all $N$). To process this formula, we convert the integral over $U$ to an integral over its $N^2$ eigenvalues $e^{i\theta_j}$. We can conveniently summarize this information in a density distribution $\rho(\theta)$ with:

$$\int_{-\pi}^{\pi} d\theta\, \rho(\theta) = 1 \tag{2.40}$$

This generates an effective action for the eigenvalues given by [4]

$$\begin{aligned} S[\rho(\theta)] = N^2 \int d\theta_1 \int d\theta_2 \rho(\theta_1)\rho(\theta_2) V(\theta_1 - \theta_2) = \\ = \frac{N^2}{2\pi} \sum_{n=1}^{\infty} |\rho_n|^2 V_n(T), \end{aligned} \tag{2.41}$$

with

$$\begin{aligned} V_n &= \frac{2\pi}{n}(1 - f(t^n, y^n, u^n, w^n)), \\ \rho_n &= \int d\theta \rho(\theta) e^{in\theta}. \end{aligned} \tag{2.42}$$

As $(1-f)$ is always positive for all allowed values of the chemical potential, it is clear that the action (2.41) is minimized by $\rho_n = 0, n > 0$; $\rho_0 = 1$. The classical value of the action vanishes on this saddle point, and the Index is given by the gaussian integral of the fluctuations of $\rho_n$ around zero. This allows us to write

$$\mathcal{I}^{WL}_{YM}\big|_{N=\infty} = \prod_{n=1}^{\infty} \frac{1}{1 - f(t^n, y^n, v^n, w^n)}. \tag{2.43}$$

If we think about the 't Hooft limit of the theory it is also interesting to compute the Index over single trace operators. This is given by

$$\begin{aligned} Z_{s.t.} = &- \sum_{r=1}^{\infty} \frac{\varphi(r)}{r} \log\left[1 - f(t^r, y^r, v^r, w^r)\right] \\ = &\frac{t^2/w}{1 - t^2/w} + \frac{vt^2}{1 - vt^2} + \frac{t^2 w/v}{1 - t^2 w/v} - \frac{t^3/y}{1 - t^3/y} - \frac{t^3 y}{1 - t^3 y} \end{aligned} \tag{2.44}$$

where $\varphi$ is the Euler Phi function and we used that $\sum_r \frac{\varphi(r)}{r} \log(1 - x^r) = \frac{-x}{1-x}$. The result (2.43) is simply the multiparticle contribution that we get from (2.44) .

Note that the action (2.41) vanished on its saddle point; as a consequence (2.43) is independent of $N$ in the large $N$ limit. This behavior, which is is reminiscent of the partition function of a large $N$ gauge theory in its confined phase, is true of (2.43) at all finite values of the chemical potential. In this respect the Index $\mathcal{I}^{YM}$ behaves in a qualitatively different manner from the free Yang Mills partition function over supersymmetric states (see the next section). This partition function displays confined behavior at large chemical potentials (analogous to low temperatures) but deconfined behavior (i.e. is of order $e^{N^2}$) at small chemical potentials (analogous to high temperature). It undergoes a sharp phase transition between these two behaviors at chemical potentials of order unity. Several recent studies of Yang Mills theory on compact manifolds have studied such phase transitions, and related them to the nucleation of black holes in bulk duals [3, 4, 19, 20, 21, 22, 23, 24]. The Index $\mathcal{I}^{WL}_{YM}$ does not undergo this phase transition, and is always in the 'confined' phase. We interpret this to mean that it never 'sees' the dual supersymmetric black hole phase.

At first sight we might think that this is a contradiction, since the black holes give a large entropy. On the other hand we are unaware of a clear argument which says that black holes should contribute to the Index. For example, it is unclear whether the Euclidean black hole geometry should contribute to the path integral that computes the Index. While the Lorentzian geometry of the black hole is completely smooth, if we compactify the Euclidean time direction with periodic boundary conditions for the spinors, then the corresponding circle shrinks to zero size at the horizon, which would represent a kind of singularity. In section 3.6 we present a mechanism for how this phenomenon (the excision of the black hole saddle point) might work in Lorentzian space.

We now present the expression for the Index in a new set of variables that are more symmetric, and for some purposes more convenient, in the study of Yang Mills theory. We will use these variables in the next section. Let us choose to parameterize charges in the subalgebra by

$$J_2,\ L_1 = E + q_1 - q_2 - q_3,\ L_2 = E + q_2 - q_1 - q_3,\ L_3 = E + q_3 - q_1 - q_2. \tag{2.45}$$

Note that $L_i$ are positive for all Yang Mills letters. A simple change of basis, yields

$$1 - f = \frac{(1 - e^{-2\gamma_1})(1 - e^{-2\gamma_2})(1 - e^{-2\gamma_3})}{\left(1 - e^{-\zeta - \gamma_1 - \gamma_2 - \gamma_3}\right)\left(1 - e^{+\zeta - \gamma_1 - \gamma_2 - \gamma_3}\right)} \tag{2.46}$$

where

$$f = \sum_{\text{letters}} (-1)^F e^{\gamma_1 L_1 + \gamma_2 L_2 + \gamma_3 L_3 + 2\zeta j_2}. \tag{2.47}$$

In section six we will write an explicit exact formulas for the Index (2.37) for $\gamma_3 = \infty$.

Further studies on the spectrum of free Yang Mills can be found in [25, 26, 27, 28].

### 2.4.2 Strong Coupling

According to the AdS/CFT correspondence, $\mathcal{N} = 4$ Yang Mills theory on $S^3$ at large $N$ and large $\lambda$ has a dual description as a weakly coupled IIB theory on the large radius $AdS_5 \times S^5$. At fixed energies in the large $N$ limit, the spectrum of the bulk dual is a gas of free gravitons, plus superpartners, on $AdS_5 \times S^5$. In this subsection, we will compute the Index $\mathcal{I}_{YM}^{WL}$ over this gas of masssless particles, and find perfect agreement with (2.43).

Note that states with energies of order one do not always dominate the partition function at chemical potentials of unit order. At small values of the chemical potential, the usual partition function of strongly coupled Yang Mills theory is dominated by black holes. However, as we have explained in the previous subsection, we do not see an argument for the black hole saddle point to contribute to the Index, and apparently it does not.

We now turn to the computation. When the spectrum of (single particle) supergravitons of Type $IIB$ supergravity compactified on $AdS_5 \times S^5$ is organized into representations of the superconformal group, we obtain representations that are built on a lowest weight state that is a $SU(2) \times SU(2)$ in the $(n, 0, 0)_{SO(6)} = (0, n, 0)_{SU(4)}$ representation of the R-symmetry group [29] . The representation with $n = 1$ is the Yang-Mills multiplet. The representation with $n = 2$ is called the 'supergraviton' representation. These representations preserve 8 of 16 supersymmetries. In the language of section 2, they are of the form **bb**. When restricted to $\Delta = 0$, they yield a representation of the subalgebra that we shall call $S_n$. $S_n$ has lowest weights $E' = n, j_2 = 0, R_2 = n, R_3 = 0$.

In the table below we explicitly list the $SU(2,1) \times SU(3)$ content of $S_n$ using the notation $[E', j_2', R_1', R_2']$ where $[E', j_2']$ specify the weight of the lowest weight state under the compact $U(1) \times SU(2)$ subgroup of $SU(2,1)$ and $[R_2', R_3']$ are Dynkin labels for $SU(3)$. This can also be found by looking at the list of Kaluza Klein modes in [29] .

For $n = 2$ we just drop the lines containing $n - 3$.

On the other hand, for $n = 1$ we have further shortening and we find Table 2.4.

Table 2.3: Content of $S_n$

| $(-\mathbf{1})^{\mathbf{F}}\,\mathbf{E}'$ | $\mathbf{J}_2'$ | $\mathbf{R}_1'$ | $\mathbf{R}_2'$ |
|---|---|---|---|
| $n$ | 0 | n | 0 |
| $-(n+\frac{1}{2})$ | $\frac{1}{2}$ | n-1 | 0 |
| $n+1$ | 0 | n-2 | 0 |
| $-(n+1)$ | 0 | n-1 | 1 |
| $n+\frac{3}{2}$ | $\frac{1}{2}$ | n-2 | 1 |
| $-(n+2)$ | 0 | n-3 | 1 |
| $n+2$ | 0 | n-1 | 0 |
| $-(n+\frac{5}{2})$ | $\frac{1}{2}$ | n-2 | 0 |
| $n+3$ | 0 | n-3 | 0 |

Table 2.4: Content of $S_1$

| $(-\mathbf{1})^{\mathbf{F}}\,\mathbf{E}'$ | $\mathbf{J}_2'$ | $\mathbf{R}_1'$ | $\mathbf{R}_2'$ |
|---|---|---|---|
| 1 | 0 | 1 | 0 |
| -$\frac{3}{2}$ | $\frac{1}{2}$ | 0 | 0 |
| -2 | 0 | 0 | 1 |
| 3 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 [10] |

The Index on single-particle states may now be calculated in a straightforward manner. The supercharacter of $S_n$ is given by

$$\chi_{Sn} = \frac{(t^{2n}\chi^{SU(3)}_{n,0}(v,w) - t^{2n+1}\chi^{SU(3)}_{n-1,0}(v,w)(y+1/y)+\ldots)}{(1-t^3y)(1-t^3/y)}. \tag{2.48}$$

The $SU(3)$ character that occurs above is described by the Weyl Character Formula. To obtain the Index, we simply need to calculate

$$\mathcal{I}_{sp} = \sum_{n=2}^{\infty}\chi_{Sn} + \chi_{S1}. \tag{2.49}$$

The sums in (2.49) are all geometric and are easily performed, yielding the single particle contribution

$$\mathcal{I}_{sp} = \frac{t^2/w}{1-t^2/w} + \frac{vt^2}{1-vt^2} + \frac{t^2w/v}{1-t^2w/v} - \frac{t^3/y}{1-t^3/y} - \frac{t^3y}{1-t^3y} \tag{2.50}$$

This matches precisely (2.44) .

From this, we may obtain the Index of the Fock Space. In particular,

$$\begin{aligned}\mathcal{I}^{WL}_{grav} &= \exp\left[\sum_n \frac{1}{n}\mathcal{I}_{sp}[t^n, v^n, w^n, y^n]\right] \\ &= \prod_{n=1}^{\infty} \frac{(1-t^{3n}/y^n)(1-t^{3n}y^n)}{(1-t^{2n}/w^n)(1-v^n t^{2n})(1-t^{2n}w^n/v^n)}\end{aligned} \tag{2.51}$$

in perfect agreement with (2.43).

Finally, let us point out that the value of the Index is the same in $\mathcal{N}=1$ marginal deformations of $\mathcal{N}=4$.[11]

## 2.5 Discussion

This chapter is based on work that was done in [14]. After the appearance of this paper, there have been several important extensions of this work; here we mention two.

The first is the computation of a similar Index in quiver gauge theories that arise when we consider D3 branes, not in flat 10 dimensional space but in $R^4 \times R^6/\Gamma$, where $\Gamma$ is some orbifolding group. This gives rise to a duality between string theory on $AdS_5 \times S^5/\Gamma$ and a supersymmetric quiver gauge theory. A computation very similar to the one above was performed in [30] and once again perfect agreement was found between the gauge theory and gravity answers.

Second, the construction above may be easily generalized to superconformal field theories in other dimensions. This was done in [31] and the results will be described

[11] These theories have the superpotential $Tr[e^{\beta}\phi_1\phi_2\phi_3 - e^{-\beta}\phi_1\phi_3\phi_2 + c(\phi_1^3+\phi_2^3+\phi_3^3)]$. If $c$ is nonzero, then we should set all chemical potentials $\gamma_i$ to be equal in the original $\mathcal{N}=4$ result, since we lose two of the $U(1)$ symmetries.

in Chapter 4

We now proceed to a study of other kinds (i.e not Indices) supersymmetric partition functions in $AdS_5/CFT_4$.

# Chapter 3

# Supersymmetric Partition Functions in $\mathbf{AdS}_5/\mathbf{CFT}_4$

## 3.1 Introduction

In this chapter, we continue our study of the $AdS_5/CFT_5$ correspondence. However, now we will focus on supersymmetric partition functions that are not protected purely by group theory.

Such partition functions are of great interest, partly because it is known that $AdS_5 \times S^5$ support BPS black holes that preserve $\frac{1}{16}$ of its supersymmetries. Unfortunately, as we saw in the last chapter, the Index over $\frac{1}{16}$ BPS states does not grow fast enough to account for the entropy of BPS black holes in $AdS_5 \times S^5$. found in [32, 33, 34].

This is not a contradiction with $AdS/CFT$; the entropy of a black hole counts all supersymmetric states with a positive sign whereas our Index counts the same states up to sign. It is possible for cancellations to ensure that the Index is much smaller than the partition function evaluated over supersymmetric states of the theory. This is certainly what happens in the free $\mathcal{N} = 4$ theory, where both quantities (the Index and the partition function) may explicitly be computed, and is presumably also the case at strong coupling.

It may well be possible to provide a weak coupling microscopic counting of the entropy of BPS black holes [32, 33, 34] in $AdS_5 \times S^5$; however such an accounting must incorporate some dynamical information about $\mathcal{N} = 4$ super Yang Mills beyond the information contained in the superconformal algebra. In this chapter we take some steps towards understanding the entropy of these black holes.

First, we note that, for large (compared to the AdS radius) black holes a naive computation of the simple partition function of BPS states in the free theory gives a formula which has similar features to the black hole answer.

Then we provide a counting of the entropy for small black holes in terms of D-branes and giant gravitons in the interior. The counting is rather similar to the one performed for the D1D5p black holes [35]. In particular, we account for the entropy of small black holes by modelling them as states in the sigma model that describes fluctuations on the moduli space of $\frac{1}{8}$ BPS giant gravitons. Related to this, we also find that in the limit of small charges, these black holes allow a near-horizon gravity description as $BTZ$ black holes in $AdS_3$. This is work that was first done in [36].

A second motivation for the study in this chapter is that the Indices described in the previous chapter do not exhaust all interesting calculable information about supersymmetric states in all superconformal field theory; in specific examples it is possible to extract more refined information about supersymmetric states by adding extra input. An explicit example where dynamical information allows us to make more progress is the computation of the chiral ring [37, 38]. In the case of $\mathcal{N} = 4$ Yang Mills theory, we write down explicit counting formulas for 1/2, 1/4 and 1/8 BPS states. The counting can be done in terms of $N$ particles in harmonic oscillator potentials. For very large charges the entropy in these states grows linearly in $N$. By taking the large $N$ limit of these partition functions we show that they display a second order phase transition which corresponds to the formation of a Bose-Einstein condensate.

## 3.2 The partition function over $\frac{1}{16}$ BPS states

In this section we will compute the partition function over BPS states that are annihilated by $Q$ and $S$ in $\mathcal{N} = 4$ Yang Mills at zero coupling and strong coupling. We perform the first computation using the free Yang Mills action, and the second computation using gravity and the AdS/CFT correspondence, together with a certain plausible assumption. Specifically, we assume that the supersymmetric density of states at large charges is dominated by the supersymmetric black holes of [32, 33, 34].

At small values of chemical potentials (when these supersymmetric partition functions are dominated by charges that are large in units of $N^2$) we find that these partition functions are qualitatively similar at weak and strong coupling but differ in detail, in these two limits. Moreover, each of these partition functions differs qualitatively from Index computed in the previous section.

Before turning to the computation, it may be useful to give a more formal description of the BPS states annihilated by $Q$ and $S$. $Q$ may formally be thought of as an exterior derivative $d$, its Hermitian conjugate $S$ is then $d^*$ and $\Delta$ is the Laplacian $dd^* + d^*d$. States with $\Delta = 0$ are harmonic forms that, according to standard arguments (see [39]), those arguments may all be reworded in the language of $Q$ and $S$ and Hilbert spaces) are in one to one correspondence with the cohomology of $Q$. $\mathcal{I}^{WL}$, the $(-1)^{degree}$ weighted partition function over this cohomology is simply the (weighted) Euler Character over this cohomology.

### 3.2.1 Partition Function at $\Delta = 0$ in free Yang Mills

Let

$$Z_{free} = \mathrm{Tr}_{\Delta=0}\left[x^{2H} e^{\mu_1 q_1 + \mu_2 q_2 + \mu_3 q_3 + 2\zeta J_2}\right] \tag{3.1}$$

where $x = e^{\frac{-\beta}{2}}$, and $q_1, q_2, q_3$ correspond to the $SO(6)$ Cartan charges (related to $R_1, R_2, R_3$ by the formulas in Appendix C of [14]). In Free Yang Mills theory this partition function is easily computed along the lines described in subsection 4.1; the final answer is given by the formula [3, 4]

$$Z = \int DU \exp\left[\sum_n \left(f_B(x^n, n\mu_i, n\zeta) + (-1)^{n+1} f_F(x^n, n\mu_i, n\zeta)\right) \frac{\mathrm{Tr}U^n \mathrm{Tr}U^{-n}}{n}\right] \tag{3.2}$$

where $U$ is a unitary matrix and the relevant 'letter partition functions' are given by

$$\begin{aligned} f_B &= \frac{(e^{\mu_1} + e^{\mu_2} + e^{\mu_3})x^2 + x^4}{(1 - x^2 e^{\zeta})(1 - x^2 e^{-\zeta})} \\ f_F &= \frac{x^3(2\cosh\zeta e^{\frac{\mu_1+\mu_2+\mu_3}{2}} + e^{\frac{\mu_1+\mu_2-\mu_3}{2}} + e^{\frac{\mu_1-\mu_2+\mu_3}{2}} + e^{\frac{-\mu_1+\mu_2+\mu_3}{2}}) - x^5 e^{\frac{\mu_1+\mu_2+\mu_3}{2}}}{(1 - x^2 e^{\zeta})(1 - x^2 e^{-\zeta})} . \end{aligned} \tag{3.3}$$

As explained in the previous section, (3.2) and (3.3) describe a partition function that undergoes a phase transition at finite values of chemical potentials. For chemical potentials such that $f_B + f_F < 1$, the integral in (3.2) is dominated by a saddle point on which $|TrU^n| = 0$ for all $n$. In this phase the partition function is obtained from the one loop integral about the saddle point (as in section 4.1) and is independent of $N$ in the large $N$ limit. The density of states in this phase grows exponentially with energy, $\rho(E) \propto e^{\beta_H E}$ where $\beta_H = -\ln(\frac{7-3\sqrt{5}}{2}) = 1.925$ and the system undergoes a phase transition when the effective inverse temperature becomes smaller than $\beta_H$ (e.g., on setting all other chemical potentials to zero, this happens at $x = e^{\frac{-\beta_H}{2}}$).

At smaller values of chemical potentials (3.2) is dominated by a new saddle point. In particular, in the limit $\zeta \ll 1$ and $\beta \ll 1$, the integral over $U$ in (3.1) is dominated by a saddle point on which $TrU^n TrU^{-n} = N^2$ for all $n$, the partition function reduces to

$$\ln Z = N^2 \sum_n \frac{1}{n} \left[ f_B(x^n, n\mu_i, n\zeta) + (-1)^{n+1} f_F(x^n, n\mu_i, n\zeta) \right]. \tag{3.4}$$

In the rest of this subsection we will, for simplicity, set $\mu_1 = \mu_2 = \mu_3 = \mu$ and thereby focus on that part of cohomology with $q_1 = q_2 = q_3 \equiv q$. The relevant letter partition functions reduce to

$$\begin{aligned} f_B &= \frac{3e^{\mu}x^2 + x^4}{(1-x^2e^{\zeta})(1-x^2e^{-\zeta})} \\ f_F &= \frac{\left(e^{\frac{3\mu}{2}}(2\cosh\zeta - x^2) + 3e^{\frac{\mu}{2}}\right)x^3}{(1-x^2e^{\zeta})(1-x^2e^{-\zeta})} \end{aligned} \tag{3.5}$$

In the limit $\beta \ll 1$, $\zeta \ll 1$ (3.4) reduces to

$$\ln Z = N^2 \frac{1}{(\beta^2 - \zeta^2)} f(\mu) \tag{3.6}$$

where

$$f(\mu) = \left(\zeta(3) + 3Pl(3, e^{\mu}) - 3Pl(3, -e^{\frac{\mu}{2}}) - Pl(3, -e^{\frac{3\mu}{2}})\right) \tag{3.7}$$

and the PolyLog function is defined by

$$Pl(m,x) = \sum_{n=1}^{\infty} \frac{x^n}{n^m} \tag{3.8}$$

This partition function describes a system with energy $E$, angular momentum $j_2$, $SO(6)$ charge $(q,q,q)$ and entropy $S$ given by[1]

$$\begin{aligned} \frac{2j_1}{N^2} \sim \frac{E}{N^2} &= 2\frac{\beta f(\mu)}{(\beta^2-\zeta^2)^2}, \\ \frac{2j_2}{N^2} &= 2\frac{\zeta f(\mu)}{(\beta^2-\zeta^2)^2} \\ \frac{q}{N^2} &= \frac{g(\mu)}{\beta^2-\zeta^2} \\ \frac{S}{N^2} &= \frac{3f(\mu)-\mu g(\mu)}{\beta^2-\zeta^2} \end{aligned} \tag{3.9}$$

where

$$g(\mu) = \frac{f'(\mu)}{3} = \left( Pl(2,e^{\mu}) - \frac{1}{2}Pl(2,-e^{\frac{\mu}{2}}) - \frac{1}{2}Pl(2,-e^{\frac{3\mu}{2}}) \right). \tag{3.10}$$

We see that for high temperatures, this partition function looks like a gas of massless particles in 2+1 dimensions. Note that in this limit $E \sim 2j_1 \gg q$.

We will sometimes be interested in the partition function with only those chemical potentials turned on that couple to charges that commute with $Q$ and $S$. This is achieved if we choose $\mu = \frac{\beta}{3}$. In the limit $\beta \ll 1$, $\zeta \ll 1$ we have $\mu \ll 1$ and the partition function and charges are given by (3.6) and (3.9) with $\mu \sim 0$; note that $f(0) = 7\zeta(3)$ and $g(0) = \frac{\pi^2}{4}$.

[1] Physically, the equations below describe Free Yang Mills theory at fixed values of charges in the limit $T \to 0$ ($T$ is the temperature). In the free theory this limit retains only supersymmetric states at all values of charges. On the other hand the black holes in [32, 33, 34] are supersymmetric in the same limit only for a subfamily of charges. See the next section for more discussion on this puzzling difference.

Note that, although the Index $\mathcal{I}^{WL}_{YM}$ and the cohomological partition function $Z_{\rm free}$ both traces over $Q$ cohomology , the final results for these two quantities in Free Yang Mills theory are rather different. For instance, at finite but small values of chemical potentials, $\ln Z_{\rm free}$ is proportional to $N^2$ (see (3.6)) while $\mathcal{I}^{WL}_{YM}$ is independent of $N$ (see (2.43)). Clearly cancellations stemming from the fluctuating sign in the definition of $\mathcal{I}^{WL}_{YM}$ cause the Index to see a smaller effective number of states. We explain this further in Section 3.6.

### 3.2.2 Cohomology at Strong Coupling: Low energies

We now turn to the study of $Q$ cohomology at strong coupling and low energies. In this limit the cohomology is simply that of the free gas of supergravitons in $AdS_5 \times S^5$, and may be evaluated following the method of subsection subsection 2.4.2. We will calculate the quantity

$$Z = \mathrm{Tr}\left[x^{2H} z^{2J_1} y^{2J_2} v^{R_2} w^{R_3}\right] \tag{3.11}$$

over the supergraviton representations restricted to states of $\Delta = 0$. We recall that the single particle states form an infinite series of short reps of the $N = 4$ superconformal algebra where the primary is a lorentz scalar with energy $n$ with R-charges $[0, n, 0]$.

The trace over single particle states may be easily calculated. The answer is [2]

$$
\begin{aligned}
Z_{\rm sp} &= \frac{{\rm num}_{\rm bos} + {\rm num}_{\rm fer}}{\rm den} \\
{\rm den} &= (1-x^2/w)(1-x^2 v)(1-x^2 w/v)(1-x^2 z/y)(1-x^2 zy) \\
{\rm num}_{\rm fer} &= x^3/y + x^3 y + x^3 z/v + v x^3 z/w + w x^3 z - 2x^5 z \\
&\quad + v x^7 z + x^7 z/w + w x^7 z/v + x^7 z^2/y + x^7 z^2 y \\
{\rm num}_{\rm bos} &= v x^2 + x^2/w + w x^2/v - x^4/v - v x^4/w - w x^4 \\
&\quad + 2x^6 + x^6 z/(yv) + v x^6 z/(wy) + w x^6 z/y - x^8 z/y \\
&\quad + x^6 zy/v + v x^6 zy/w + w x^6 zy - x^8 zy + x^4 z^2 + x^{10} z^2
\end{aligned}
\tag{3.13}
$$

The full (multi particle) partition function over supersymmetric states may be obtained by applying the formulas of Bose and Fermi statistics to (3.13).

Special limits of (3.13) will be of interest in the next section. For instance, the limit $z \to 0$ focuses on states with $\Delta = 0$ and $j_1 = 0$, i.e. (1/8) BPS states. In this limit (3.13) becomes

$$
\begin{aligned}
Z^{1/8}_{\text{bos}-\text{sp}} &= \frac{1-(1-x^2/w)(1-vx^2)(1-wx^2/v)+x^6}{(1-x^2/w)(1-vx^2)(1-wx^2)} \\
Z^{1/8}_{\text{fer}-\text{sp}} &= \frac{x^3(y+1/y)}{(1-x^2/w)(1-vx^2)(1-wx^2/v)}
\end{aligned}
\tag{3.14}
$$

[2] In the notation of the previous subsection, with $y = e^{\zeta}$,

$$
\begin{aligned}
Z^{\rm res}_{sp} &= Tr\left[x^{2H} y^{2J_2} u^{2\sum q_i}\right] = \frac{{\rm num}^{\rm res}_{\rm bos} + {\rm num}^{\rm res}_{\rm fer}}{{\rm den}^{\rm res}} \\
{\rm den} &= (1-x^2u^2)^3(1-x^2/y)(1-x^2 y) \\
{\rm num}_{\rm fer} &= 3ux^3 - 2u^3x^5 + 3u^5x^7 + (u^3x^3)/y + (u^3x^7)/y \\
&\quad + u^3x^3y + u^3x^7y \\
{\rm num}_{\rm bos} &= 3u^2x^2 + x^4 - 3u^4x^4 + 2u^6x^6 + u^6x^{10} \\
&\quad + (3u^4x^6)/y - (u^6x^8)/y + 3u^4x^6y - u^6x^8y
\end{aligned}
\tag{3.12}
$$

In terms of the $\gamma_i$ variables introduced at the end of subsection 4.1

$$\begin{aligned} Z^{1/8}_{\text{bos-sp}} &= \frac{1-(1-e^{-2\gamma_1})(1-e^{-2\gamma_2})(1-e^{-2\gamma_3})+e^{-2(\gamma_1+\gamma_2+\gamma_3)}}{(1-e^{-2\gamma_1})(1-e^{-2\gamma_2})(1-e^{-2\gamma_3})} \\ Z^{1/8}_{\text{fer-sp}} &= \frac{e^{-\gamma_1-\gamma_2-\gamma_3}\left[e^{\zeta}+e^{-\zeta}\right]}{(1-e^{-2\gamma_1})(1-e^{-2\gamma_2})(1-e^{-2\gamma_3})} \end{aligned} \tag{3.15}$$

Applying the formulas for Bose and Fermi statistics, it is now easy to see that the partition function over the Fock space, including multi-particle states, is given by

$$Z^{1/8}(\zeta,\gamma_1,\gamma_2,\gamma_3) = \prod_{n,m,r=0}^{\infty} \frac{\prod_{s=\pm1}(1+e^{s\zeta}e^{-(2n+1)\gamma_1-(2m+1)\gamma_2-(2r+1)\gamma_3})}{(1-e^{-2n\gamma_1-2m\gamma_2-2r\gamma_3})(1-e^{-(2n+2)\gamma_1-(2m+2)\gamma_2-(2m+2)\gamma_3})} \tag{3.16}$$

Finally, in order to compute the rate of growth of the cohomological density of states with respect to energy, we set $z, y, v, w \to 1$. This gives the "blind" single particle partition function which is

$$\begin{aligned} Z^{\text{bl}}_{\text{bos-sp}} &= \frac{x^2(3-2x^2+8x^4-2x^6+x^8)}{(1-x^2)^5} \\ Z^{\text{bl}}_{\text{fer-sp}} &= \frac{x^3(5-2x^2+5x^4)}{(1-x^2)^5} \end{aligned} \tag{3.17}$$

The full partition function is given by

$$Z^{\text{bl}} = \exp\left[\sum_n \frac{Z^{\text{bl}}_{\text{bos-sp}}(x^n)+(-1)^{n+1}Z^{\text{bl}}_{\text{fer-sp}}(x^n)}{n}\right] \tag{3.18}$$

Let

$$x = e^{\frac{-\beta}{2}}. \tag{3.19}$$

At small $\beta$ this partition function is approximately given by

$$\ln Z = \frac{63\zeta(6)}{4\beta^5}. \tag{3.20}$$

It follows that the entropy as a function of energy is given by

$$S(E) = h\log n(E) \sim \frac{6}{5}\left(\frac{315\zeta(6)}{4}\right)^{\frac{1}{6}} E^{5/6} \approx 2.49E^{5/6}. \tag{3.21}$$

Note that this is slower than the exponential growth of the same quantity at zero coupling.

### 3.2.3 Cohomology at Strong Coupling: High Energies

Gutowski and Reall [32, 33], and Chong, Cvetic, Lu and Pope [34] have found a set of supersymmetric black holes in global $AdS_5 \times S^5$, that are annihilated by the supercharges $Q$ and $S$. These black holes presumably dominate the supersymmetric cohomology at energies of order $N^2$ or larger. In this subsection we will translate the thermodynamics of these supersymmetric black holes to gauge theory language, and compare the results with the free cohomology of subsection 5.1.

Restricting to black holes with $q_1 = q_2 = q_3 = q$ these solutions constitute a two parameter set of solutions, with thermodynamic charges, translated to Yang Mills Language via the AdS/CFT dictionary[3],

$$\begin{aligned}
\frac{E}{N^2} &= (a+b)\frac{[(1-a)(1-b)+(1+a)(1+b)(2-a-b)]}{2(1-a)^2(1-b)^2} \\
\frac{j_1+j_2}{N^2} &= \frac{(a+b)(2a+b+ab)}{2(1-a)^2(1-b)} \\
\frac{j_1-j_2}{N^2} &= \frac{(a+b)(a+2b+ab)}{2(1-a)(1-b)^2} \\
\frac{q}{N^2} &= \frac{(a+b)}{2(1-a)(1-b)} \\
\frac{S}{N^2} &= \frac{\pi(a+b)\sqrt{a+b+ab}}{(1-a)(1-b)}.
\end{aligned} \tag{3.22}$$

Setting $a = 1-(\beta'+\zeta')$ and $b = 1-(\beta'-\zeta')$, and assuming $\beta' \ll 1$, $\zeta' \ll 1$, (3.22)

[3] We have set $g = 1$ in [34] and set $E_{CFT} = E_{Chong\ et\ al}/G_5$, where $G_5 = G_{N5}/R^3_{AdS}$ is the value of Newton's constant in units where the $AdS_5$ radius is set to one. $S_{here} = S_{Chong\ et\ al}/G_5$. For $\mathcal{N} = 4$ Yang Mills we have $G_5 = \frac{\pi}{2N^2}$. To convert formulas in [32, 33] simply set this value for the five dimensional Newton constant in their expressions.

reduces to

$$\begin{aligned}\frac{2j_1}{N^2} \sim \frac{E}{N^2} &\sim \frac{8\beta'}{(\beta'^2-\zeta'^2)^2}\\ \frac{2j_2}{N^2} &\sim \frac{-8\zeta'}{(\beta'^2-\zeta'^2)^2}\\ \frac{q}{N^2} &\sim \frac{1}{\beta'^2-\zeta'^2}\\ \frac{S}{N^2} &\sim \frac{2\sqrt{3}\pi}{\beta'^2-\zeta'^2}\end{aligned} \tag{3.23}$$

Equations (3.22) and (3.9) have some clear similarities[4] in form, but also have one important qualitative difference. (3.9) has one additional parameter absent in (3.22). After setting the three $SO(6)$ charges equal the $Q$ cohomology is parametrized by 3 charges, whereas only a two parameter set of supersymmetric black hole solutions are available.

We should emphasize that in the generic, non BPS, situation black hole solutions are available for all values of the 4 parameters $q, j_2, j_1$ and $E$ [34] . It is thus possible to continuously lower the black hole energy while keeping $q, j_2$ and $j_1$ fixed at arbitrary values. The temperature of the black hole decreases as we lower its energy, until it eventually goes to zero at a minimum energy. However the extremal black hole thus obtained is supersymmetric (its mass saturates the supersymmetric bound) only on a 2 dimensional surface in the 3 dimensional space of charges parameterized by $q, j_2$ and $j_1$. For every other combination of charges the zero temperature black holes are not supersymmetric (their mass is larger than the BPS bound). We are unsure how this should be interpreted[5]. It is possible that, for other combinations of charges, the

[4] This observation has also been made by H. Reall and R. Roiban.

[5] Note that our Index $\mathcal{I}_{YM}^{WL}$, when specialized to states with $q_1 = q_2 = q_3$, also depends on two rather than 3 parameters.

cohomology is captured by as yet undiscovered supersymmetric black solutions.

In order to compare the cohomologies in (3.9) and (3.22) in more detail, we choose $\mu$ in (3.9) so that the equations for $E/N^2$ and $q/N^2$ in (3.9) and (3.22) become identical (after a rescaling of $\beta'$ and $\zeta'$). This is achieved provided that

$$f(\mu_c)^2 = 16 g(\mu_c)^3 \tag{3.24}$$

This equation is easy to solve numerically. We find $\mu_c = -0.50366 \pm .00001$ and that $f(\mu_c) = 5.7765$, $g(\mu_c) = 1.2776$. Plugging in $\mu = \mu_c$ into the entropy formula in (3.9) we then find

$$\frac{S_{\text{Field}}}{S_{\text{Black-Hole}}} = \frac{3\frac{f(\mu)}{g(\mu)} - \mu}{2\sqrt{3}\pi} = 1.2927 \tag{3.25}$$

Another way to compare (3.9) and (3.22) is the following. First notice that the charge $q$ is much smaller than the energy in this limit, $q \ll E$. Let us set $\mu = \beta/3$ which is the value that we have in the Index (though we do not insert the $(-1)^F$ we have in the Index). Since we are taking the limit where $\beta$ is small we can evaluate $f$ in (3.9) at zero, $f(0) = 7\zeta(3)$. By comparing the energies and entropies in (3.9) and (3.22) and writing the free energy as $E = N^2 c\beta^{-3}$, where $c$ is a "central charge" that measures the number of degrees of freedom. Then we can compute

$$\frac{c_{\text{gravity}}}{c_{\text{free-field-theory}}} = \frac{\pi^3}{14\zeta(3)3^{3/2}} \sim 0.35458... \tag{3.26}$$

It is comforting that this value is lower than one since we expect that interactions would remove BPS states rather than adding new ones. A similar qualitative agreement between the weak and strong coupling was observed between the high temperature limit of uncharged black holes and the free Yang Mills theory [40] , where the ratio (3.26) is 3/4. Note that for $\mu = \beta/3$ we can approximate $g$ in the expression for the charge in (3.9) by $g(0) \neq 0$. This agrees qualitatively with the expression coming from black holes.

## 3.3 Small Supersymmetric Black Holes in $AdS_5$ as Giant Gravitons

Let us set $j_2 = 0$ or $a = b$ in (3.22). We then expand the resulting expression for low values of $a$.

$$\begin{aligned} \frac{E}{N^2} &\sim 3a \sim 3\frac{q}{N^2} \\ \frac{j_1}{N^2} &\sim 3a^2 \\ \frac{S}{N^2} &\sim 2\pi\sqrt{2}a^{3/2}. \end{aligned} \tag{3.27}$$

In this limit, these small supersymmetric black holes are interesting for two reasons. First, it is possible to count the entropy of these black holes using D-branes in $AdS$. This is not the same problem as counting them in the field theory, but perhaps these results might be a good hint for the kind of states that we should look at in the field theory.

Second, in this limit, it turns out that on *lifting* these black holes to solutions of $10D$ supergravity, one may see that they have a near-horizon limit that looks very much like a BTZ black hole in $AdS_3$. While, this may seem to provide a justification for the counting done above, it also leads to a puzzle that we will discuss below.

### 3.3.1 D brane counting of the small black hole entropy

The limit we have taken above focussed on black holes with charges that are much larger than $N$ but much smaller than $N^2$. In this regime, the supersymmetric sector of the Hilbert space of the theory is presumably well described by probe $D3$ branes moving about in the bulk.

The idea of this subsection is to count the number of such probe configurations that could lead to states that, macroscopically, look like the small black hole desribed above.

First, let us review the description of $\frac{1}{8}$ BPS giant-gravitons in $AdS_5 \times S^5$ that was given by Mikhailov [41].

For this purpose, it is convenient to embed the $S^5$ of the $AdS_5$ in $C^3$. We take an arbitrary holomorphic 2-complex dimensional surface in $C^3$ and we intersect it with $\sum |z_i|^2 = 1$. This gives a 3-real dimensional surface on $S^5$. A $D3$ brane wrapping this surface is what we call a 'giant graviton'; such a configuration preserves $\frac{1}{8}$ of the bulk supersymmetries.

The black-holes above carry charge corresponding a particular $U(1) \subset SU(4)$. Geometrically, this $U(1)$ can be thought of as coming from a KK reduction of a particular $S^1$ in the $S^5$. Translations along this $S^1$ correspond to simultaneous rotations of the $z_i$ above by a phase: $z_i \to e^{i\alpha} z_i$ (where $\alpha$ is some phase). For the giant graviton to have finite charge under this $U(1)$, it must wrap this $S^1$ an integral number of times. Now, geometrically, wrapping the $S^1$ means that the 'giant graviton' configuration is invariant under shifts along the $S^1$.

The configurations that have this property are those where the holomorphic surface in $C^3$ is invariant under simultaneous rotations of the $z_i$ above. The most general such surface is given by the *roots* of a homogeneous polynomial of degree $n$:

$$\sum_{n_1+n_2+n_3=n} C_{n_1,n_2,n_3} z_1^{n_1} z_2^{n_2} z_3^{n_3} = 0 \tag{3.28}$$

It is easy to see that this 2-surface intersects the sphere $|z_i|^2$ n-times. So, a giant graviton that is wrapped on this 2-surface has energy $nN$, and also $U(1)$ charge $nN$. Notice, that in general there are $\frac{n(n-1)}{2} \sim n^2$ distinct *complex* coefficients that specify the polynomial above.

Now, we think of a configuration where the coefficients $C_{n_1,n_2,n_3}$ vary slowly as a function of the common angle $z_1, z_2, z_3$. We can think of this as a 'left-moving' fluctuation in the sigma model on the space of all possible coefficients $C_{n_1,n_2,n_3}$. The central charge of this sigma model is $n^2+\frac{n^2}{2} = \frac{3n^2}{2}$ (each complex coefficient has central charge 2 and its superpartner has central charge 1). Apart from its 'rest energy', the giant graviton now has some additional energy, say $p$, that comes from this variation. The number of configurations with energy $p$ is given by Cardy's formula as

$$N_{\text{config}} = e^{2\pi\sqrt{\frac{3n^2 p}{12}}} \tag{3.29}$$

The total energy however is $E_{\text{total}} = nN + p$.

Now, there is a *family* of sigma models, each parameterized by a particular value of $n$. We can consider all of these in a grand canonical partition function. The entropy is dominated by that value of $n$ that gives the largest number in (3.29) above. This is obtained by maximizing $N_{\text{config}}$ subject to the constraint that the total energy be a constant. This maximization is easily done. We find that, at the maximum

$$\begin{aligned} n = \frac{2E_{\text{total}}}{3N}, \quad p = \frac{E_{\text{total}}}{3} \\ N_{\text{config}} = e^{2\pi \frac{E_{\text{total}}}{3}^{\frac{3}{2}}} \end{aligned} \tag{3.30}$$

Comparing with (3.27), we see that the entropy is almost correct but off by a factor of 2. However, our insight that these black holes can be realized as states in a $1+1$ sigma model is important; in fact, in the next subsection, we will find that, indeed, in the near-horizon limit the geometry of these black holes looks like a BTZ black hole in AdS$_3$.

## 3.4 Near Horizon Geometry of Supersymmetric Black Holes in AdS$_5$

In this subsection we will investigate the near-horizon geometry of supersymmetric black holes in AdS$_5$. This subsection is based on work done in [36]. The near-horizon geometry of extremal black holes in AdS$_5$ was also explored in [42].

We will find that, in fact, in this limit the geometry begins to looks like AdS$_3$. This provides a justification for the sigma model that we constructed to explain black holes in the previous section; however, as we shall see, some puzzles remain.

Once again we focus on the limit (3.27). These small black holes are parameterized by a single parameter. Here, we will find it convenient to take that parameter to be $R_0$ which is consistent with the notation of [32]. In this subsection, we will also restore factors of the $AdS$ radius $\ell$. In terms of these parameter, the charge and energy of this black hole are given by the following formulae.

$$q = \frac{\pi \ell {R_0}^2}{4G_5}\left(1+\frac{{R_0}^2}{2\ell^2}\right) \ , \quad j_1 = \frac{3\pi {R_0}^4}{8G_5\ell}\left(1+\frac{2{R_0}^2}{3\ell^2}\right) \ , \tag{3.31}$$

where $G_5$ is the 5 dimensional Newton constant, related to the type IIB one by

$$\frac{1}{G_5} = \frac{vol(S^5)}{G_{10}} = \frac{\pi^3\ell^5}{G_{10}} \ . \tag{3.32}$$

The Energy of the black hole is given as

$$E = \frac{3\pi {R_0}^2}{4G_5}\left(1+\frac{3{R_0}^2}{2\ell^2}+\frac{2{R_0}^4}{3\ell^4}\right) \ . \tag{3.33}$$

The above mass and charges satisfy the BPS relation

$$E = \frac{3q+2j_1}{\ell} \tag{3.34}$$

The full solution is given by the following 5 dimensional metric and gauge field (see the section 3.2 of [33], whose simplified version for equal charges is presented here):

$$\begin{aligned} {ds_5}^2 &= -f^2dt^2 - 2f^2\omega dt\sigma_3 + f^{-1}g^{-1}dR^2 \\ &+ \frac{R^2}{4}\left(f^{-1}({\sigma_1}^2+{\sigma_2}^2) + f^2h{\sigma_3}^2\right) \end{aligned} \tag{3.35}$$

$$A \equiv A^1 = A^2 = A^3 = f dt + \sigma_3\left(\frac{R^2+2{R_0}^2}{2\ell} + f\omega\right) \tag{3.36}$$

where

$$\begin{aligned} f &= \left(1+\frac{{R_0}^2}{R^2}\right)^{-1} \ , \quad \omega = -\frac{1}{2\ell}\left(R^2+3{R_0}^2+\frac{3{R_0}^4}{2R^2}\right) \ , \\ g &= 1+\frac{3{R_0}^2}{\ell^2}+\frac{R^2}{\ell^2} \ , \\ h &\equiv f^{-3}g - \frac{4}{R^2}\omega^2 \\ &= 1+\frac{3{R_0}^2}{R^2}+\frac{{R_0}^4}{R^4}\left(3+\frac{{R_0}^2}{\ell^2}\right)+\frac{{R_0}^6}{R^6}\left(1+\frac{3{R_0}^2}{\ell^2}-\frac{9{R_0}^2}{4\ell^2}\right) \end{aligned} \tag{3.37}$$

and $\ell$ is the radius of $AdS_5$ at asymptotic infinity ($R \to \infty$). The horizon is at $R = 0$. $\sigma^i$ ($i = 1, 2, 3$) are the right-invariant 1-forms, i.e., invariant under the $SU(2)_R$ of $SO(4) = SU(2)_L \times SU(2)_R$. In terms of the Hopf coordinates $(\theta, \phi; \psi)$, these one-forms are:

$$\begin{aligned} \sigma_3 &= d\psi + \cos\theta d\phi \\ \sigma_1 &= \sin\psi d\theta - \cos\psi \sin\theta d\phi \\ \sigma_2 &= \cos\psi d\theta + \sin\psi \sin\theta d\phi \ . \end{aligned} \tag{3.38}$$

Note that the above time coordinate $t$, we have used above, is not associated with the energy but rather with the isometry generated by $\sim \bar{\epsilon}\gamma_\mu\epsilon$, where $\epsilon$ is the Killing denotes the Killing spinor preserved by the geometry.

### 3.4.1 Uplifting the Solution

The uplift ansatz for the above 5 dimensional supergravity solutions to type IIB supergravity can be found in the section 2.1 of [43]. In general, if all $Q_1$, $Q_2$ and $Q_3$ are different, the uplifted solution has $U(1)^3 \subset SO(6)$ isometry. The former is generated by the Cartan elements of $SO(6)$. When all three charges are equal ($Q_1 = Q_2 = Q_3$), one may consider the solution with $A^1 = A^2 = A^3$, which is the one presented in the previous section. Then the uplifted solution would preserve the $U(1) \times SU(3) \subset SO(6)$ isometry of $S^5$. This is similar to the fact that one can take the solution to have $U(1)_L \times SU(2)_R \subset SO(4)$ isometry for $J_1 = J_2$ case.

The notation of [43] is related to ours as follows.

$$\begin{aligned} g \text{ of } [43] &\rightarrow \frac{1}{\ell}\ , \\ A^i \text{ of } [43] &\rightarrow A^I, \\ X_i \text{ of } [43] &\rightarrow X^I \text{ of } [33]\ , \end{aligned}$$

where $i = 1, 2, 3$, $I = 1, 2, 3$, and in our case ($Q_1 = Q_2 = Q_3$) all scalars are constant $X^I = 1$.

Here we consider the uplifted metric, which is basically eqn.(2.1) (and eqn.(2.7)) of [43] with $X_i = 1$ ($i = 1, 2, 3$). For $A^1 = A^2 = A^3 \equiv A$ case, it is natural to take advantage of the Hopf-fibration of $S^5$: a circle fibration over a $CP^2$ base. The $U(3)$ isometry mentioned above comes from that of $CP^2$. The ten dimensional metric is given as

$$ds_{10}^2 = {ds_5}^2 + \ell^2 \left( ds^2_{CP^2} + (d\Psi + V + \frac{1}{\ell}A)^2 \right) \equiv {ds_5}^2 + \ell^2 \left( ds^2_{CP^2} + (\Sigma + \frac{1}{\ell}A)^2 \right)\ , \tag{3.39}$$

where ${ds_5}^2$ and $A$ are given by (3.35) and (3.36), respectively. $\Psi$ is an angle with period $\Psi \sim \Psi + 2\pi$. The $CP^2$ metric and $V$ are the standard Fubini-Study ones. For instance, the metric is

$$ds^2_{CP^2} = \frac{dz^a d\bar{z}^a}{1 + \bar{z}^b z^b} - \frac{(\bar{z}^a dz^a)(z^b d\bar{z}^b)}{1 + \bar{z}^c z^c} = dz^a d\bar{z}^b\ \partial_a \bar{\partial}_b\ \log(1 + \bar{z}^c z^c) \tag{3.40}$$

where the summation over $a, b, c = 1, 2$ is assumed. The volume of $CP^2$ associated with this metric is given as $vol(CP^2) = \frac{\pi^2}{2}$. The 1-form $V$, in a suitable gauge, is given as

$$V = -\frac{i}{2} \frac{\bar{z}^a dz^a}{1 + \bar{z}^b z^b} + c.c. \tag{3.41}$$

Inserting the expressions (3.35) and (3.36) and expanding, one obtains

$$
\begin{aligned}
ds_{10}^2 &= -f^2dt^2 - 2f^2\omega dt\sigma_3 + f^{-1}g^{-1}dR^2 + \frac{R^2}{4}\left(f^{-1}({\sigma_1}^2+{\sigma_2}^2) + f^2h{\sigma_3}^2\right) \\
&\quad +2\ell\Sigma\left(fdt + \sigma_3\left(\frac{R^2+2{R_0}^2}{2\ell}+f\omega\right)\right) + \left(fdt+\sigma_3\left(\frac{R^2+2{R_0}^2}{2\ell}+f\omega\right)\right)^2 \\
&\quad +\ell^2\Sigma^2 + \ell^2 ds_{CP^2}^2 \\
&= 2fdt\left(\ell\Sigma + \sigma_3\left(\frac{R^2+2{R_0}^2}{2\ell}\right)\right) + \ell^2\Sigma^2 + 2\ell\,\Sigma\sigma_3\left(\frac{R^2+2{R_0}^2}{2\ell}+f\omega\right) \\
&\quad +{\sigma_3}^2\left(\frac{R^2}{4}f^{-1}g + 2f\omega\left(\frac{R^2+2{R_0}^2}{2\ell}\right) + \left(\frac{R^2+2{R_0}^2}{2\ell}\right)^2\right) \\
&\quad +f^{-1}g^{-1}dR^2 + \ell^2ds_{CP^2}^2 + \frac{R^2}{4}f^{-1}({\sigma_1}^2+{\sigma_2}^2)\ ,
\end{aligned}
\tag{3.42}
$$

which we rearrange as

$$
\begin{aligned}
ds_{10}^2 &= 2\left(1+\frac{{R_0}^2}{R^2}\right)^{-1}dt\left(\ell\Sigma+\sigma_3\left(\frac{R^2+2{R_0}^2}{2\ell}\right)\right) \\
&\quad +\alpha(R)\left(\ell\Sigma+\sigma_3\left(\frac{R^2+2{R_0}^2}{2\ell}\right)\right)^2 + \beta(R)\left(\ell\Sigma+\gamma(R)\sigma_3\right)^2 \\
&\quad +\frac{\ell^2({R_0}^2+R^2)}{\ell^2+3{R_0}^2+R^2}\,\frac{dR^2}{R^2} + \ell^2ds_{CP^2}^2 + \frac{{R_0}^2+R^2}{4}\,\left({\sigma_1}^2+{\sigma_2}^2\right)
\end{aligned}
\tag{3.43}
$$

where

$$
\begin{aligned}
\alpha(R) &= 1-\beta(R) \\
\beta(R) &= \frac{4}{R^2}f^3g^{-1}\omega^2 = \frac{(\frac{3}{2}{R_0}^4+3{R_0}^2R^2+R^4)^2}{({R_0}^2+R^2)^3(\ell^2+3{R_0}^2+R^2)} \\
\gamma(R) &= \frac{R^2+2{R_0}^2}{2\ell} + \frac{R^2}{4}f^{-2}g\omega^{-1} = -\ell\ \frac{{R_0}^4 - \frac{{R_0}^4}{2\ell^2}R^2 + R^4\left(1-\frac{3{R_0}^2}{\ell^2}\right)}{3{R_0}^4+6{R_0}^2R^2+2R^4}\ .
\end{aligned}
\tag{3.44}
$$

The rearranged form (3.43) will be useful when we take the near-horizon limit.

The near-horizon limit $R^2 \to \epsilon R^2$ and $t \to \frac{t}{\epsilon}$ with $\epsilon \to 0$. In this limit we have

$$
\alpha(R) \to \frac{\ell^2+\frac{3}{4}{R_0}^2}{\ell^2+3{R_0}^2}\ ,\quad \beta(R) \to \frac{\frac{9}{4}{R_0}^2}{\ell^2+3{R_0}^2}\ ,\quad \gamma(R) \to -\frac{\ell}{3}\ .
\tag{3.45}
$$

The metric in this limit is

$$\begin{aligned} ds_{10}^{\ 2} \quad \rightarrow \quad & 2\frac{R^2}{R_0^{\ 2}}\, dt \left(\ell\Sigma + \frac{R_0^{\ 2}}{\ell}\sigma_3\right) + \frac{\ell^2 + \frac{3}{4}R_0^{\ 2}}{\ell^2 + 3R_0^{\ 2}} \left(\ell\Sigma + \frac{R_0^{\ 2}}{\ell}\sigma_3\right)^2 + \frac{\ell^2 R_0^{\ 2}}{\ell^2 + 3R_0^{\ 2}}\, \frac{dR^2}{R^2} \\ & + \ell^2 ds_{CP^2}^{\ 2} + \frac{R_0^{\ 2}}{4}\, \left(\sigma_1^{\ 2} + \sigma_2^{\ 2}\right) + \frac{\frac{9}{4}R_0^{\ 2}}{\ell^2 + 3R_0^{\ 2}} \left(\ell\Sigma - \frac{\ell}{3}\, \sigma_3\right)^2 . \end{aligned} \tag{3.46}$$

Note that the angle appearing in the last term $\propto (3\Sigma - \sigma_3)^2$ is

$$\phi^1 + \phi^2 + \phi^3 - \varphi^1 - \varphi^2 \ , \tag{3.47}$$

where the five angles are conjugate to the charges $Q_1$, $Q_2$, $Q_3$, $J_1$ and $J_2$, respectively. Let us define $r \equiv \frac{R^2}{R_0^{\ 2}}$.

### 3.4.2 Parameters and Charges

The metric (3.46) is that of a near extremal BTZ black hole fibred over a base. We will first try to understand the base; to that end we will permit ourselves a small digression.

#### $S^3$ and $S^5$ as Hopf Fibrations

Consider the $S^3 \times S^5$ that lies inside $AdS_5 \times S^5$. The $S^3/S^5$ may respectively be thought of as Hopff fibrations over $S^2$ and $CP^2$ respectively. Concretely we have

$$ds_3^2 = ds_2^2 + (d\psi + \cos\theta d\phi)^2 \tag{3.48}$$

$$ds_5^2 = ds_{CP^2} + (d\Psi + V)^2 \tag{3.49}$$

$$\tag{3.50}$$

where $\cos\theta d\phi$ and $V$ respectively are, respectively, the Fubini Study one forms on $CP^1$ and $CP^2$ respectively, and $\psi$ and $\Psi$ are each phases of periodicity $2\pi$ each.

We work this out in detail for $S^3$. Let $z_1 = \cos\theta/2e^{i\alpha_1}$ and $z_2 = \sin\theta/2e^{i\alpha_2}$ on $C^2$. Clearly $\theta$ varies from $0-\pi$ while ?$\alpha_i$ vary from $0-2\pi$. The base $S^2$ is parameterized by $\theta$ and $\phi = \alpha_1 - \alpha_2$. We choose $\psi = \alpha_1 + \alpha_2$ as our fibre coordinate. The identifications $(\psi, \chi) \sim (\psi + 4\pi, \phi)$ and $(\psi, \chi) \sim (\psi + 2\pi, \phi + 2\pi)$ generate the lattice of $\alpha_i$ identifications. Here $\psi$ represents the fibre (note that one of the identifications make the fiber coordinate periodic in a manner that makes no reference to the base point - this is a consistency check for the interpretation as a fibration with our choice of base). [6]

Upon working out the metric in these new coordinates we find

$$ds^2 = \frac{1}{4}\left[\left(d\theta^2 + \sin^2\theta d\phi^2\right) + \left(d\psi + \cos\theta d\phi\right)^2\right] \tag{3.51}$$

We conclude that unit radius $S^3$ may be written as a $2\pi$ valued fibration over a radius half 2 sphere (though the range of the coordinate $\psi$ is $4\pi$, the length of the circle it parameterizes is $2\pi$.

In the same way we may embed $S^5$ inside $C^3$. Let $z_i = r_i e^{i\beta_i}$. The coordinate $\Psi$ may be chosen to be $\frac{\beta_1+\beta_2+\beta_3}{3}$; with this choice $\Psi$ has periodicity $2\pi$. Setting $\beta_1 = \beta_2 = \beta_3$ ensures we move along the fiber; clearly the length of the corresponding circle, in the metric $dz_i d\bar{z}_i$ is $2\pi$. All points along the $(1,1,1)$ vector (in $\beta_i$ space) map to the same base point as the base is $CP^2$.

Finally let us compute the volumes of the bases obtained via this procedure. We have $\omega_{d-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$; note in particular that $\omega_1 = 2\pi$, $\omega_2 = 4\pi$, $\omega_3 = 2\pi^2$, $\omega_4 = \frac{8}{3}\pi^2$, $\omega_5 = \pi^3$. We find that the volume of the base $S^2$ is $2\pi^2/2\pi = \pi$, which is the correct

[6]The precise nature of the coordinate $\psi$ along the fibre is not important - one could add any function of the base to it, at the price of modifying the one form that apperas in (3.46).

volume for a radius 1/2 2 sphere. On the other hand the volume of $CP^2$ base is $\pi^2/2$.

**The volume of the base**

The base in the metric (3.46) is

$$\ell^2 ds^2_{CP^2} + \frac{{R_0}^2}{4}\left({\sigma_1}^2 + {\sigma_2}^2\right) + \frac{\frac{9}{4}{R_0}^2}{\ell^2 + 3{R_0}^2}\left(\ell\Sigma - \frac{\ell}{3}\,\sigma_3\right)^2$$

where $\Sigma \sim \Psi$ and $\sigma_3 \sim \psi$. We will now compute the periodicities of the base coordinate $3\Sigma - \sigma_3$ and the fiber coordinate $\Sigma + \frac{R_0^2}{l^2}\sigma_3$.

In the space $(\Sigma, \sigma_3/2)$ we may choose the identification lattice vectors as $(2,3)$ and $(m,k)$ where $2k - 3m = 1$. The first vector is chosen to be constant along the base, and the second vector then completes the unit cell. The periodicity of the base coordinate is then $(3m-2k)2\pi$ while the periodicity of the fibre coordinate is $(4\pi)(1+3\frac{R_0^2}{l^2})$.

It follows that the volume of the base is $l^4\pi^2/2 \times R_0^2\pi \times R_0\pi \times \sqrt{1+3\frac{R_0^2}{l^2}} = \frac{l^4 R_0^3\pi^4}{2} \times \sqrt{1+3\frac{R_0^2}{l^2}}$.

**The Central Charge**

In (3.46) we have $l_3 = \frac{R_0}{\sqrt{1+3\frac{R_0^2}{l^2}}}$ It follows that

$$l_3 V_7 = \frac{l^4 R^4 \pi^4}{2} \tag{3.52}$$

In conventions in which the action takes the form $\mathcal{L} = \frac{1}{16\pi G_3}\sqrt{g}R + \ldots$ the $AdS_3$ central charge is $c = \frac{3l_3}{2G_3}$ . Using the same conventions for the 10 dimensional action we find $c = \frac{3l_3V_7}{2G_{10}}$ or

$$c = \frac{3\pi^4}{4}\frac{R_0^4 l^4}{G_{10}} = \frac{3}{2}\frac{R_0^4}{l^4} \times \frac{\pi l^3}{2G_5} \tag{3.53}$$

We now use the relation

$$\frac{2G_5}{l^3\pi} = 1/N^2 \tag{3.54}$$

to find

$$c = 6N^2\left(\frac{R_0^2}{2l^2}\right)^2 \tag{3.55}$$

Using

$$\frac{q}{N^2} = \frac{R_0^2}{2l^2}(1 + \frac{R_0^2}{2l^2}) \tag{3.56}$$

the central charge is easily expressed as a function of $q$; in the limit of small charge we find.

$$c \approx \frac{6q^2}{N^2} \tag{3.57}$$

**The BTZ Mass**

The non dimensionalized mass or angular momentum of the extremal BTZ black hole is given by

$$L_0 = \frac{R_0^2}{4G_3 l_3} \tag{3.58}$$

Using

$$r_0^2 = 4l^2 \times \frac{1 + \frac{3}{4}\frac{R_0^2}{l^2}}{1 + 3\frac{R_0^2}{l^2}} \tag{3.59}$$

$$l_3 = \frac{R_0}{\sqrt{1 + 3\frac{R_0^2}{l^2}}} \tag{3.60}$$

$$G_3 = \frac{G_5 \pi^3 l^5 l_3}{V_7 l_3} \tag{3.61}$$

together with (3.52) we find

$$L_0 = 2\frac{R_0^2}{2l^2} \times \frac{\pi l^3}{2G_5} \times (1 + \frac{3}{4}\frac{R_0^2}{l^2}) = 2N^2\frac{R_0^2}{2l^2} \times (1 + \frac{3}{4}\frac{R_0^2}{l^2}) \approx 2N^2\frac{q}{N^2} = 2q \qquad (3.62)$$

where the approximation works in the limit of small charge.

**Entropy**

The entropy that follows from Cardy's formula is

$$\begin{aligned} S &= 2\pi\sqrt{\frac{cL_0}{6}} && (3.63) \\ &= 2\pi N\sqrt{2\left(\frac{R_0^2}{2l^2}\right)^3\left(1 + \frac{3}{4}\frac{R_0^2}{l^2}\right)} && (3.64) \\ &\approx 2\sqrt{2}\pi\frac{q^{\frac{3}{2}}}{N} && (3.65) \\ & && (3.66) \end{aligned}$$

where, once again, the approximation is accurate in the limit of small charges. The entropy in (3.63) is in precise agreement with the formulas of Gutowski and Reall.

### 3.4.3 Discussion

In the previous two sections we have presented two derivations of the entropy of small black holes in $AdS_5$. The first discussion above, in section (3.3), led us to believe that small black holes in $AdS_5$ should be describable by a $1+1$ sigma model. In section (3.4), we verified that in the near-horizon, these black holes did look like $AdS_3$ fibered on a base. This suggests that when if we look at excitations that are 'near' the states corresponding to $\frac{1}{16}$ small black holes, the theory of these excitations is describable by a $1+1$ dimensional CFT. This would be like 'AdS/CFT within AdS/CFT'!

However, there are puzzles and details to be cleared up in this picture. First, one needs to understand whether the sigma model description in section (3.3) is justified. While counting the entropy of D1-D5 black holes, we work in a limit where the compact manifold is much smaller than the circle on which the CFT is defined. This is evidently not the case in the sigma model that we presented; the $CP^2$ is the same size as the $S^1$ on which the CFT is defined. This may account for the discrepancy between (3.30) and (3.27). Second, in section (3.4), the $\mathrm{AdS}_3$ geometry that we found is fibered on a base. This makes the question of dimensional reduction somewhat tricky.

Finally, it would be interesting if one could find an explicit operator counting of these small black holes. The two discussions we have presented above may be helpful in this regard.

## 3.5 Partition Functions over $\frac{1}{2}$, $\frac{1}{4}^{th}$ and $\frac{1}{8}^{th}$ BPS States

In this section we will study the partition function over $\frac{1}{8}^{th}$, a quarter and half BPS supersymmetric states in $\mathcal{N}=4$ Yang Mills. We will compute these partition functions in free Yang Mills, at weak coupling using naive classical formulas, and at strong coupling using the AdS/CFT correspondence. In the case of quarter and $\frac{1}{8}^{th}$ BPS states, our free and weak coupling partition functions are discontinuously different. However the weak coupling and strong coupling partition functions agree with each other (see [44] for an explanation).

It is possible that something similar will turn out to be true for the $\frac{1}{16}^{th}$ cohomology (see [2] for a possible mechanism). This makes the enumeration of the weakly coupled $Q$ cohomology an important problem. We hope to return to this problem in the near future.

### 3.5.1 Enumeration of $\frac{1}{8}^{th}$, quarter and half BPS Cohomology

In this subsection we will enumerate operators in the anti-chiral ring, i.e. operators that are annihilated by $Q^{\alpha 1}$, with $\alpha = \pm\frac{1}{2}$, and their Hermitian conjugates (these are the charges we called $Q$ and $Q'$ in previous sections [7]). All such states have $\Delta = 0$ and $j_1 = 0$. It is not possible to isolate the contribution of these states to $\mathcal{I}_{YM}$ (note the Index lacks a chemical potential that couples to $j_1$); nonetheless we will be able to use dynamical information to count these states below.

This enumeration is easily performed in the free theory. Only the letters $X$, $Y$, $Z$, $\psi_{0,\pm,+++}$ (see Table 2.2) and no derivatives contribute in this limit. We will denote these letters by $\bar{\Phi}_i$ ($i = 1\ldots 3$) and $\bar{W}_{\dot{\alpha}}$ ($\dot{\alpha} = \pm$) below. Note that these letters all have $j_1 = 0$ and $E = q_1 + q_2 + q_3$. The partition function

$$Z_{cr-free} = Tr \exp\left[\sum_i \mu_i q_i + 2\zeta j_2\right] \tag{3.67}$$

is given by the expression on the RHS of (3.2) with

$$f_B = \sum_{i=1}^{3} e^{\mu_i}, \quad f_F = 2\cosh\zeta e^{\frac{\mu_1+\mu_2+\mu_3}{2}}. \tag{3.68}$$

[7] If we had chosen states annihilated by $\bar{Q}_1^{\dot{\alpha}}$ we would have obtained the chiral ring.

Note that $1 - f_B - f_F$ is positive at small $\mu_i$ but negative at large $\mu_i$. We conclude that the partition function (3.67) undergoes the phase transition described in [3, 4] at finite values of the chemical potentials, and that its logarithm evaluates to an expression of order $N^2$ at small $\mu_i$.

We now turn to the weakly interacting theory. As explained in [37, 44], at any nonzero coupling no matter how small, the set of supersymmetric operators is given by all gauge invariant anti-chiral fields $\bar{\Phi}_i, \bar{W}_{\dot{\alpha}}$ modulo relations $[\bar{\Phi}_i, \bar{\Phi}_j] = [\bar{\Phi}_i, \bar{W}_{\dot{\alpha}}] = 0$ and $\{\bar{W}_{\dot{\alpha}}, \bar{W}_{\dot{\beta}}\} = 0$ (the first of these follows from $\partial_{\bar{\Phi}_i}\bar{W} = 0$ where $\bar{W}$ is the superpotential). In general there can be corrections to these relations (see [44] ). We assume that such corrections do not change the number of elements in the ring. In fact, if we go to the Coulomb branch of $\mathcal{N} = 4$ we get a $U(1)^N$ theory with no corrections at the level of the two derivative action. The chiral primary operators at a generic point of this moduli space are the same as all the operators that we are going to count.

It is now easy to enumerate the states in the chiral ring. The relations in the previous paragraph ensure that all the basic letters commute or anticommute, and so may be simultaneously diagonalized, so we must enumerate all distinct polynomials of traces of diagonal letters. This is the same thing as enumerating all polynomials of the $3N$ bosonic and $2N$ fermionic eigenvalues that are invariant under the permutation group $S_N$. We may now formally substitute the eigenvalues $\bar{\phi}_i^f$ and $\bar{W}_{\dot{\alpha}}^f$ $(f = 1 \dots N)$ with bosonic and fermionic creation operators $a_i^f$ and $w_{\dot{\alpha}}^f$; upon acting on the vacuum these produce states in the Hilbert space of $N$ particles, each of which propagates in the potential of a 3 dimensional bosonic and a 2 dimensional fermionic oscillator. The permutation symmetry ensures that the particles are identical bosons or fermions depending on how many fermionic oscillators are excited. As a consequence we conclude that the cohomological partition function is given by the coefficient of $p^N$ in

$$\begin{aligned} Z_{1/8}(p,\gamma_1,\gamma_2,\gamma_3,\zeta) &= \sum_{N=0}^{\infty} p^N Z_N(\gamma_1,\gamma_2,\gamma_3,\zeta) \\ &= \prod_{n,m,r=0}^{\infty} \frac{\prod_{s=\pm1}(1+p\,e^{s\zeta}e^{-(2n+1)\gamma_1-(2m+1)\gamma_2-(2r+1)\gamma_3})}{(1-p\,e^{-n2\gamma_1-m2\gamma_2-r2\gamma_3})(1-p\,e^{-(2n+2)\gamma_1-(2m+2)\gamma_2-(2m+2)\gamma_3})} \end{aligned} \tag{3.69}$$

Further discussion on these 1/8 BPS states can be found in [38] .

We may now, specialize both the free and the interacting cohomologies listed above to $\frac{1}{4}^{th}$ BPS cohomology by taking the limit $\gamma_3 \to \infty$. The only letters that contribute in this limit are $\bar{\Phi}_1$ and $\bar{\Phi}_2$ ($X, Y$ of Table 2.2). The final result for the interacting cohomology may be written as

$$Z_{1/4}(p,\gamma_1,\gamma_2) = \sum_{N=0}^{\infty} p^N Z_N(\gamma_1,\gamma_2) = \frac{1}{\prod_{n,m=0}^{\infty}(1-p\,e^{-n2\gamma_1-m2\gamma_2})} \tag{3.70}$$

For a more explicit construction of 1/4 BPS operators see [45] and references therein.

It is instructive to compare the $\gamma_3 \to \infty$ limit (3.70) of (3.69) to the same limit of the partition function over $Q$ cohomology of the previous section that also simplifies dramatically in this limit.The only letters that contribute in this limit are $X, Y, \Psi_{+,++-}$ (where the Indices refer to $j_1, q_1, q_2, q_3$ charges). Further, it is easy to verify that $Q\Psi_{+,++-} \propto [X,Y]$. As a consequence the matrices $X$ and $Y$ should commute and may be diagonalized; furthermore the matrix $\psi$ must also be diagonal (so that $Q$ anihilates it). The cohomology in this limit is thus given by the partition function of $N$ particles in a 2 bosonic and one fermionic dimensional harmonic oscillator.

$$Z = \sum_N p^N Tr[y^{2J_1} e^{-\gamma_i L_i}] = \prod_{n,m\geq 0} \frac{(1 + pye^{-2(n+1)\gamma_1 - 2(m+1)\gamma_2})}{(1 - pe^{-2n\gamma_1 - 2m\gamma_2})} \tag{3.71}$$

The Index $\mathcal{I}^{WL}$ over this cohomology is then computed by setting $y = -1$. At this special value, terms in the numerator with values $m, n$ cancel against terms in the denominator with $m+1, n+1$ leaving only

$$\begin{aligned} \mathcal{I}^{WL}_{YM} &= \sum_N \mathcal{I}^{WL}_{YM}(N) = \sum_N p^N Tr_N[(-1)^F e^{-\gamma_i L_i}] \\ &= \frac{1}{(1-p)\prod_{n=1}^{\infty}(1 - pe^{-n2\gamma_1})(1 - pe^{-n2\gamma_2})} \end{aligned} \tag{3.72}$$

This is an exact formula for the $\gamma_3 \to \infty$ limit of the Index $\mathcal{I}^{WL}_{YM}$. Multiplying it with $(1-p)$ and setting $p$ to unity, we recover the large $N$ result (2.43) (at $\gamma_3 = \infty$).

It is also possible to further specialize (3.70) to the half BPS cohomology (of states annihilated by supercharges with $q_1 = \frac{1}{2}$) by taking the further limit $\gamma_2 \to \infty$ to obtain

$$Z_{1/2}(p, \gamma_1) = \sum_{N=0}^{\infty} p^N Z_N(\gamma_1) = \frac{1}{\prod_{n=0}^{\infty}(1 - p\, e^{-n2\gamma_1})} \tag{3.73}$$

Note that the free half BPS cohomology, interacting half BPS cohomology and the $\gamma_2, \gamma_3 \to \infty$ limit of $\mathcal{I}^{WL}_{YM}$ all coincide. On the other hand the free quarter BPS cohomology sees many more states than the interacting quarter BPS cohomology which, in turn, sees a larger effective number of states than the $\gamma_3 \to \infty$ limit of the Index. The last quantity, the Index, receives contributions from $\bar{\Phi}_1$, $\bar{\Phi}_2$ and $\psi_{+,0;++-}$, which are all the states in Table 2.2 which have $L_3 = 0$. This Index sees a smaller number of states as a consequence of cancellations involving the presence of the fermion $\psi_{+,0;++-}$. Again, the $\frac{1}{8}^{th}$ BPS free cohomology sees more states than the interacting cohomology, which in turn sees more states than the Index, with no restrictions on chemical potentials. More explicitly, we can see that for very large charges, or very small chemical potentials the entropy of (3.73) is that of $N$ harmonic oscillators, which correspond basically to the eigenvalues. Similarly, (3.70), and (3.69) give the entropy of $2N$ and $3N$ harmonic oscillators respectively. All these entropies are basically linear in $N$ in the large $N$ limit. The intuitive reason is that the matrices commute, and so do not taking advantage of the full non-abelian structure of the theory.

### 3.5.2 Large $N$ limits and Phase Transitions

In this subsection we will study the large $N$ limit of the partition functions (3.69), (3.70), (3.73). We will first briefly consider the limit $N \to \infty$ at fixed values of the chemical potentials, and show that in this limit these partition functions reproduce the supergravity answers (3.15) . We will then turn to large $N$ limits in which the chemical potentials scale with $N$. We find that the formulas for 1/4 and 1/8 BPS states lead to large $N$ phase transitions. This phase transition is the Bose-Einstein condensation of the lowest mode, the ground state of the harmonic oscillators we had in the previous subsection.

In the $N \to \infty$ and fixed chemical potential the partition functions (3.73), (3.70), (3.69) , become independent of $N$. This limit is most easily evaluated by multiplying the partition functions by $(1-p)$ [8] and setting $p = 1$. The entropy then grows as a gas of massless particles in one, two and three dimensions respectively.

[8]This step cancels the divergent contribution of the ground state of the harmonic oscillator in this limit. We will have a lot more to say about this below.

For half BPS states we have [46]

$$Z_{1/2}(\gamma_1) = \frac{1}{\prod_{n=0}^{\infty}(1 - e^{-n2\gamma_1})} \tag{3.74}$$

Clearly, in the large $N$ limit, (3.74) may be thought of as the multiparticle partition function for a system of bosons with

$$Z_{1/2-sp} = \sum_{n=1}^{\infty} e^{-2n\gamma_1} = \frac{1}{1 - e^{-2\gamma_1}} - 1; \tag{3.75}$$

note that (3.75) is the same as the supergravity result (3.15) in the limits $\gamma_2 \to \infty$, $\gamma_1 \to \infty$. Similarly the large $N$ limit of (3.70) is the multiparticle partition function for a system of bosons whose single particle partition function is

$$Z_{1/4-sp} = \sum_{n,m} e^{-2n\gamma_1 - 2m\gamma_2} = \frac{1}{(1 - e^{-2\gamma_1})(1 - e^{-2\gamma_2})} - 1, \tag{3.76}$$

which is the same as (3.15) in the limit $\gamma_3 \to \infty$. In a similar fashion, in the large $N$ limit of (3.69) is precisely the multiparticle partition function (3.16) – a system of bosons and fermions whose single particle partition functions are given by (3.15).

We now turn to large $N$ limits of these partition functions in which we will allow the chemical potentials to scale with $N$. Let us start with the 1/2 BPS case, and set $\gamma_1 = \gamma$. This case does not have a phase transition. We write

$$\log Z(\gamma, p) = -\sum_n \log(1 - p\, e^{-2n\gamma}) \sim -\frac{1}{2\gamma}\int_0^{\infty} dx \log(1 - p\, e^{-x}) \tag{3.77}$$

We can first solve for $p$ by writing

$$N = p\,\partial_p \log Z = \frac{1}{2\gamma}\int_0^{\infty} dx \frac{p\, e^{-x}}{1 - p\, e^{-x}} = -\frac{1}{2\gamma}\log(1 - p) \tag{3.78}$$

We can now write $\tilde{\beta} \equiv N2\gamma$. Then (3.78) is independent of $N$ and it has a solution for all values of $\tilde{\beta}$ . We can then write the partition function as

$$\log Z_N(\gamma) = N\left\{\frac{1}{\tilde{\beta}}\int_0^{\infty} dx \log[1 - (1 - e^{-\tilde{\beta}})e^{-x}] - \log(1 - e^{-\tilde{\beta}})\right\} \tag{3.79}$$

We see that this formula is of order $N$. There is no transition as a function of $\tilde{\beta}$. For large values of $\tilde{\beta} \ll 1$, it turns out that (3.79) is independent of $N$ when expressed in terms of $\gamma$. This can be most easily seen by setting $p = 1$ in (3.77) . As expected the change in behavior happens at a temperature $(2\gamma)^{-1} \sim N$ which is when the trace relations start being important. For very small $\tilde{\beta}$ we find that (3.79) becomes $\log Z_N \sim N[-\log \tilde{\beta} + 1]$, which captures the large temperature behavior of $N$ harmonic oscillators plus an $1/N!$ statistical factor.

Let us now consider 1/4 BPS states. Let us set $\gamma_1 = \gamma_2 = \gamma$. For sufficiently large temperatures we approximate the partition function as

$$\log Z(\beta, p) = \sum_{n_1, n_2} - \log(1 - pe^{-(n_1+n_2)2\gamma}) \sim \frac{1}{(2\gamma)^2} \int_0^\infty dxx[-\log(1 - pe^{-x})] \quad (3.80)$$

Now we find a new feature when we compute

$$N = \frac{1}{(2\gamma)^2} \int dxx \frac{pe^{-x}}{1 - pe^{-x}} = \frac{1}{(2\gamma)^2} Pl[2, p] \quad (3.81)$$

where $Pl[2, p]$ is the PolyLog function. Now we see that for the lowest value of the chemical potential, $p = 1$, we get

$$N_{max} = \frac{1}{(2\gamma)^2} \frac{\pi^2}{6} \quad (3.82)$$

Defining $\tilde{\beta} \equiv 2\gamma\sqrt{N}$ we see that there is a critical temperature, $\tilde{\beta}_c^2 = \frac{\pi^2}{6}$, at which there is a phase transition obtained by setting $N_{max} = N$ in (3.82) . At temperatures smaller than this temperature we have a Bose-Einstein condensation of the ground state of the harmonic oscillator. In this regime the free energy $Z_N(\beta)$ is given by (3.80) with $p = 1$. For higher temperatures we are supposed to solve for $p$ using (3.81) and then insert it in (3.80) to compute the free energy. We get

$$\log Z_N(\tilde{\beta}') = N \left\{ \frac{1}{\tilde{\beta}^2} \int_0^\infty dxx[-\log(1 - p(\tilde{\beta})e^{-x})] - \log p(\tilde{\beta}) \right\} \quad (3.83)$$

where $p(\tilde{\beta})$ is the solution to (3.81) . Then for large temperatures we have $\log Z_N \sim N[-\log\tilde{\beta}^2 + 1]$ which captures the entropy of $N$ 2-dimensional harmonic oscillators plus the $1/N!$ statistical factor. It is possible to see that at $\tilde{\beta}_c$ we have a second order phase transition.

One can find similar results for the 1/8 BPS states. We set $\gamma_i = \gamma$. In this case the rescaled temperature is given by $\tilde{\beta}' = 2\gamma N^{1/3}$. The results are similar. For low temperatures the answer is independent of $N$ and for high temperatures we have a free energy which is linear in $N$ and is a function of the rescaled temperature $\tilde{\beta}'$. Again there is a second order phase transition corresponding to the Bose-Einstein condensation of ground state of the harmonic oscillator. If we think of these harmonic oscillators as arising from D3 branes wrapping the $S^3$, then we could think of this condensation as responsible for the fact that the $S^3$ is contractible, in the spirit of the transition in [47]. It would be nice to see if this can be made more precise.

## 3.6 Comparing the Cohomological Partition Function and the Index

Let the number of states with charges $J_1, J_2, L_i$ be given by $e^{S(J_1,J_2,L_i)}$. Then

$$\begin{aligned} Z_{\text{free}} &= \sum_{J_1,J_2,L_i} \exp\left[S(J_1,J_2,L_i) - \sum_i \gamma_i L_i - 2\zeta J_2\right] \\ \mathcal{I}^{WL}_{YM} &= \sum_{J_1,J_2,L_i} \exp\left[S(J_1,J_2,L_i) - \sum_i \gamma_i L_i - 2\zeta J_2\right](-1)^{2(J_1+J_2)} \end{aligned} \tag{3.84}$$

where we have set all chemical potentials that couple to charges outside $SU(2,1|3)$ to zero in $Z_{\text{free}}$. Let

$$\exp\left[N^2 S_{\text{eff}}(\tilde{j}_1, \gamma_i)\right] = \sum_{J_2, L_i} \exp\left[S(J_1, J_2, L_i) - \sum_i \gamma_i L_i - 2\zeta J_2\right]. \tag{3.85}$$

where $\tilde{j}_1 \equiv J_1/N^2 \gg 1$ and $\gamma_i \ll 1$. Let us assume that $S_{\text{eff}}$ is independent of $N$ in the large $N$ limit. We certainly have this property in the free theory, and we expect it in the interacting $\mathcal{N}=4$ theory, but it does not have to hold for every theory. We can then rewrite (3.84) as

$$\begin{aligned} Z_{\text{free}} &= \sum_{J_1} \exp\left[N^2 S_{\text{eff}}(\tilde{j}_1, \zeta, \gamma_i)\right] \\ \mathcal{I}^{WL}_{YM} &= \sum_{J_1} \exp\left[N^2 \left\{S_{\text{eff}}(\tilde{j}_1, \zeta + \pi i, \gamma_i) + 2i\pi \tilde{j}_1\right\}\right] \end{aligned} \tag{3.86}$$

Let us assume that that at fixed values of $\zeta, \gamma_i$ has a maximum at $\tilde{j} = a(\theta, \gamma_i)$ and that

$$\begin{aligned} S_{\text{eff}}(a+\delta, \zeta, \gamma_i) &\approx S_0 - 2b^2\delta^2 \\ S_0 &= S_{\text{eff}}(a, \zeta, \gamma_i). \end{aligned} \tag{3.87}$$

The contribution of this saddle point to the partition function in the first line of (3.86) is easily estimated[9] by

$$Z_{\text{free}} \approx \sqrt{\frac{2\pi}{b^2 N^2}} \exp\left[N^2 S_0\right]. \tag{3.88}$$

An estimation of the Index in the second line of (3.86) is a more delicate task as the summand changes by large values over integer spacings. To proceed we will

[9]For instance one could convert the sum into an integral using the Euler McLaurin formula [48] and approximate the integral using saddle points. A more careful estimate may be obtained by Poisson resumming, see the next paragraph.

*assume* that $S_{\text{eff}}(j_1, \zeta, \gamma_i)$ is a continuous function; i.e. that it does not evaluate to discontinuously different answers for integral and half integral values of $J_1$. This is a nontrivial assumption, which we believe to be true for free Yang Mills theory, but will not always be true in every theory. Under this assumption we will now estimate the contribution of the saddle point at $\tilde{j}_1 = a$ to the Index by

$$\begin{aligned} \mathcal{I}_{YM}^{WL} &= e^{N^2 S_0} \sum_{m=-\infty}^{\infty} \exp\left[-\frac{b^2 m^2}{2N^2} + \pi i m\right] \\ &= e^{N^2 S_0} \sum_{k=-\infty}^{\infty} \sqrt{\frac{2\pi}{b^2 N^2}} \exp\left[\frac{N^2(2\pi)^2}{2b^2}(k-\frac{1}{2})^2\right] \\ &\approx 2\sqrt{\frac{2\pi}{b^2 N^2}} \exp\left[N^2(S_0 - \frac{\pi^2}{2b^2})\right] \end{aligned} \tag{3.89}$$

where we have used the Poisson resummation formula to go from the first to the second line of (3.89).

Note that the contribution of the saddle point at $\tilde{j}_1 = a$ to the Index is supressed compared to its contribution to the partition function. Moreover, if $S_0 < \pi^2/2b^2$, the contribution of this saddle point is formally of order $e^{-aN^2}$; which means that the neighborhood of the saddle point does not contribute significantly to the Index in the large $N$ limit; the Index receives its dominant contributions from other regions of the summation domain. An estimation from formulas of (3.6), (3.9) puts us in this regime

As a toy example of the suppression described in the last two paragraphs, consider the two identities

$$\begin{aligned} Z &= (2+1)^N = \sum_k 2^k \frac{N!}{k!(N-k)!} \\ I &= (2-1)^N = \sum_k 2^k \frac{N!}{k!(N-k)!}(-1)^{N-k}. \end{aligned} \tag{3.90}$$

The summation over $k$ in the first of (3.90) may be approximated by the integral

$$\int_{x=0}^{1} e^{N \ln \frac{2^x}{x^x (1-x)^{1-x}}}, \tag{3.91}$$

which localizes around the saddle point value $x^s = \frac{2}{3}$ at large $N$, yielding $Z = 3^N$. The contribution to $I$ from this saddle point, on the other hand, is proportional to $e^{N(\ln 3 - \frac{\pi^2}{3})}$, and so is utterly negligible, consistent with the fact that $I$ evaluates to unity. [10]

## 3.7 Discussion

This chapter is based on the second half of the paper [14]. Some progress has been made in calculating supersymmetric partition functions in $AdS/CFT$ after the appearance of this paper.

First, the conjecture made here for the exact $\frac{1}{N}, \frac{1}{8}$ BPS partition function was verified in [49]. The authors of this paper considered $\frac{1}{8}$ BPS giant gravitons in $AdS_5$ and showed that by exactly quantizing this moduli space, they could reproduce the partition function (3.69). This work itself was followed up by other papers that used similar geometric quantization techniques to obtain exact results for the chiral ring in theories with Sasaki-Einstein duals [50, 51].

Then, the conjecture made here for exact $\frac{1}{16}$ BPS partition function in the strong coupling infinite $N$ limit was verified in [52]. The authors of this paper used recently

[10] Actually, a computation very similar to this toy example explains why the Index grows more slowly that exponentially with energy in the 'low temperature phase' (while the cohomological partition function displays exponential growth in the same phase). The number of states that contribute at energy $E$ to the Index is given by the coefficient of $x^E$ in (2.43). This is given by a multinomial expansion. When we weight the sum with $(-1)^F$, the multinomial sum stops growing exponentially just like (3.90) above. Hence, the Index never goes through a Hagedorn like transition.

developed 'integrability' techniques to obtain a partition function that agreed with (3.13).

Finally, since the appearance of this work there have been several attempts to calculate the exact entropy of $\frac{1}{16}$ BPS black holes in $\mathrm{AdS}_5$ [53, 54]. However, this problem remains an important unsolved problem. Apart from the problem of counting the entropy, one would also like to explain why supersymmetric $\frac{1}{16}$ black holes in $\mathrm{AdS}_5$ have only 4 parameters and not 5. An explanation of this curious fact should go a long way towards enhancing our understanding of the AdS/CFT duality.

# Chapter 4

# Indices for Superconformal Field Theories in $3, 5$ and $6$ Dimensions

## 4.1 Introduction

In the previous two chapters, we have studied an Index for 4 dimensional superconformal field theories. Such a construction may be easily generalized to field theories in $3, 5, 6$ dimensions. Superconformal algebras do not exist for $d > 6$. On the other hand, in 2 dimensions, conformal symmetry is enhanced; the Index in this case is rather famous and called the 'elliptic genus'. We will study this Index in the next 2 chapters.

Superconformal algebras in $d = 3, 6$ also arise in the worldvolume theory on $M2$ and $M5$ branes respectively. In the simplest examples, these theories are dual to $M$ theory on $\mathrm{AdS}_4 \times S^7$ and $\mathrm{AdS}_7 \times S^4$ respectively. In this chapter we will calculate our Index in supergravity on these backgrounds. The worldvolume theory for a large number of coincident $M2$ and $M5$ branes is not known; however, the Index we calculate in this chapter should match with the Index for those theories, whatever they are. This will serve as an important check on these theories, whenever they are found.

In this chapter, we follow the pattern of Chapter 2. We perform a study of superconformal algebras in $d = 3, 5$ and 6 and use our results to provide a complete classification of all superconformal Indices in these dimensions. In each of these cases, we also provide a trace formula that, when evaluated in a superconformal field theory, may be used to extract all these superconformal Indices. This is the analogue of the trace formula described in [2] for the Witten Index. Thus the Witten Index we define in this chapter constitutes the most general superconformal Index in $d = 3, 5, 6$.

We then proceed to compute our superconformal Witten Index for specific superconformal field theories. We first perform this computation for the superconformal field theories on the world volume of $N$ M2 and $N$ M5 branes, at $N = 1$ (using field theory) and at $N = \infty$ (using the dual supergravity description). We find that our Index has significant cancellations compared to the simple partition function over supersymmetric states. In each case, the density of states in the Index grows slower in comparison to the supersymmetric entropy. We also compute our Index for some of the Chern Simons superconformal field theories recently analyzed by Gaiotto and Yin [55]; and find that, in some cases, this Index undergoes a large $N$ phase transition as a function of chemical potentials.

This chapter is divided into 3 self-contained parts. Superconformal algebras in $d = 3$ are analyzed in Section 4.2, in $d = 6$ are discussed in Section 4.3 and in $d = 5$

are discussed in 4.4. In each section, we describe the relevant algebra and its unitary representations. We then discuss short representations and enumerate all possible ways in which short representations can pair up into long representations. We use this enumeration to provide, in each dimension, an exhaustive list of all Indices that are protected by group theory alone. We also provide a trace formula for a Witten type Index that may be evaluated via a path integral. These Indices count states that are annihilated by a particular supercharge. We discuss how the Witten Index may be expanded out in characters of the subalgebra of the superconformal algebra that commutes with this supercharge. The coefficients of these characters in the Witten Index are nothing but the Indices mentioned above.

In $d = 3$, we evaluate our Index in three different theories: (a) Supergravity on $AdS_4 \times S^7$ (b) the worldvolume theory of a single $M2$ brane and (c) the Chern-Simons matter theories recently discussed in [55]. In $d = 6$, we evaluate our Index in supergravity on $AdS_7 \times S^4$ and in the worldvolume theory of a single $M5$ brane.

## 4.2 d=3

### 4.2.1 The Superconformal Algebra and its Unitary Representations

The bosonic subalgebra of the $d = 3$ superconformal algebra is $SO(3, 2) \times SO(n)$ (the conformal algebra times the R symmetry algebra). The anticommuting generators in this algebra may be divided into the generators of supersymmetry ($Q$) and the generators of superconformal symmetries ($S$). Supersymmetry generators transform in the vector representation of the R-symmetry group $SO(n)$,[1] have charge half under dilatations (the $SO(2)$ factor of the compact $SO(3) \times SO(2) \in SO(3,2)$) and are spinors under the $SO(3)$ factor of the same decomposition. Superconformal generators $S_i^\mu = (Q_\mu^i)^\dagger$ transform in the spinor representation of $SO(3)$, have scaling dimension (dilatation charge) $(-\frac{1}{2})$, and also transform in the vector representation of the R-symmetry group. In our notation for supersymmetry generators $i$ is an $SO(3)$ spinor Index while $\mu$ is an $R$ symmetry vector Index.

We pause to remind the reader of the structure of the commutation relations and irreducible unitary representations of the $d = 3$ superconformal algebra (see [7] and references therein ). As usual, the anticommutator between two supersymmetries is proportional to momentum times an $R$ symmetry delta function, and the anticommutator between two superconformal generators is obtained by taking the Hermitian conjugate of these relations. The most interesting relationship in the algebra is the anticommutator between $Q$ and $S$. Schematically

$$\{S_i^\mu, Q_\nu^j\} \sim \delta_\nu^\mu T_i^j - \delta_i^j M_\nu^\mu$$

Here $T_i^j$ are the $U(2) \sim SO(3) \times SO(2)$ generators and $M_\nu^\mu$ are the $SO(n)$ generators.

Irreducible unitary lowest energy representations of this algebra possess a distinguished set of lowest energy states called primary states. Primary states have the lowest value of $\epsilon_0$ – the eigenvalue of the dilatation (or energy) operator – of all states in their representation. They transform in irreducible representation of

[1] In the literature on the worldvolume theory of the $M2$ brane, the supercharges are taken to transform in a spinor of $SO(8)$. This is consistent with the statement above, because for $n = 8$, the vector and spinor representations are related by a triality flip and a change of basis takes one to the other.

$SO(3) \times SO(n)$, and are annihilated by all superconformal generators and special conformal generators.[2]

Primary states are special because all other states in the unitary (always infinite dimensional) representation may be obtained by acting on the primary with the generators of supersymmetry and momentum. For a primary with energy $\epsilon_0$, a state obtained by the action of $k$ different $Q$ s on the primary has energy $\epsilon_0 + \frac{k}{2}$, and is said to be a state at the $k^{\text{th}}$ level in the representation. The energy, $SO(3)$ highest weight (denoted by $j = 0, \frac{1}{2}, 1 \ldots$) and the R-symmetry highest weights $(h_1, h_2 \ldots h_{[n/2]})$ [3] of primary states form a complete set of labels for the entire representation in question.

Any irreducible representation of the superconformal algebra may be decomposed into a finite number of distinct irreducible representations of the conformal algebra. The latter are labeled by their own primary states, which have a definite lowest energy and transform in a given irreducible representation of $SO(3)$. The state content of an irreducible representation of the superconformal algebra is completely specified by the quantum numbers of its constituent conformal primaries.

As we have mentioned in the introduction, the superconformal algebra possesses special short or BPS representations which we will now explore in more detail. Consider a representation of the algebra, whose primary transforms in the spin $j$ representation of $SO(3)$ and in the $SO(n)$ representation labeled by highest weights $\{h_i\}$ $i = 1, \cdots, \left[\frac{n}{2}\right]$. We normalize primary states to have unit norm. The superconformal algebra – plus the Hermiticity relation $(Q^i_\mu)^\dagger = S^\mu_i$ – completely determines

[2]i.e. all generators of negative scaling dimension.

[3]$h_i$ are eigenvalues under rotations in orthogonal 2 planes in $R^n$. Thus, for instance, $\{h_i\} = (1, 0, 0, ..0)$ in the vector representation

the inner products between any two states in the representation. All states in an unitary representation must have positive norm: however this requirement is not algebraically automatic, and, in fact imposes a nontrivial restriction on the quantum numbers of primary states. This restriction takes the form $\epsilon_0 \geq f(j, h_i)$ as we will now explain.[4]

Let us first consider descendant states, at level one, of a representation whose primary has $SO(3)$ and $SO(n)$ quantum numbers $j, (h_1 \ldots h_{[n/2]})$. It is easy to compute the norm of all such states by using the commutation relations of the algebra. As long as $j \neq 0$ it turns out that the level one states with lowest norm transform in in the spin $j - \frac{1}{2}$ representation of the conformal group and in the $(h_1 + 1, \{h_i\})$ $i = 2, \cdots, \left[\frac{n}{2}\right]$ representation of $SO(n)$ [7]. The highest weight state in this representation may be written explicitly as (see [12])

$$|Zn_1\rangle = A_1^-|h.w\rangle \equiv \left(Q_1^{-\frac{1}{2}} - Q_1^{\frac{1}{2}} J_- \left(\frac{1}{2J_z}\right)\right)|h.w\rangle \tag{4.1}$$

where $J_-$ denotes the spin lowering operator of $SO(3)$ and $Q_1^{\pm\frac{1}{2}}$ are supersymmetry operators with $j = \pm\frac{1}{2}$ and $(h_1, h_2, \ldots h_{[n/2]}) = (1, 0, \ldots, 0)$. Here $|h.w\rangle$ is a highest weight state with energy $\epsilon_0$, $SU(2)$ charge $j$ and $SO(n)$ charge $(h_1, h_2, \ldots, h_{[n/2]})$. The norm of this state is easily computed and is given by,

$$\langle Zn_1|Zn_1\rangle = \left(1 + \frac{1}{2j}\right)(\epsilon_0 - j - h_1 - 1) \tag{4.2}$$

It follows that the non negativity of norms of states at level one (and so the unitarity of the representation) requires that the charges of the primary should satisfy

$$\epsilon_0 \geq j + h_1 + 1 \tag{4.3}$$

[4]These techniques have been used in the investigation of unitarity bounds for conformal and superconformal algebras in [5, 56, 10, 11, 7, 12].

For $j \neq 0$ this inequality turns out to be the necessary and sufficient condition for a representation to be unitary.

When the primary saturates the bound (4.3) the representation possess zero norm states: however it turns out to be consistent to define a truncated representation by simply deleting all zero norm states. This procedure yields a physically acceptable representation whose quantum numbers saturate (4.3). This truncated representation is unitary (has only positive norms) but has fewer states than the generic representation, and so is said to be 'short' or BPS.

The set of zero norm states we had to delete, in order to obtain the BPS representation described above, themselves transform in a representation of the superconformal algebra. This representation is labeled by the primary state $|Zn_1\rangle$ (see (4.1)).

Let us now turn to the special case $j = 0$. In this case $|Zn_1\rangle$ is ill defined and does not exist; no states with its quantum numbers occur at level one. The states of lowest norm at level one transform in the spin half $SO(3)$ representation, and have $SO(n)$ highest weights $h_1' = h_1 + 1, \{h_i\}\ i = 2, \cdots, \frac{n}{2}$. The highest weight state in this representation is $|Zn_2\rangle = A_1^+|h.w.\rangle \equiv Q_1^{\frac{1}{2}}|h.w\rangle$. The norm of this state is $(\epsilon_0 - h_1)$. Unitarity thus imposes the constraint $\epsilon_0 \geq h_1$. However, in this case, this condition is necessary but not sufficient to ensure unitarity, as we now explain.

As we have remarked above, the state $|Zn_1\rangle = A_1^-|h.w\rangle$ is ill defined when $j = 0$. However $|s_2\rangle = \left(A_1^+ A_1^-\right)|h.w\rangle = Q_1^{\frac{1}{2}} Q_1^{-\frac{1}{2}}|h.w\rangle$ is well defined even in this situation (when $j = 0$). The norm of this state is easily computed and is given by,[5]

$$\langle s_2 | s_2 \rangle = (\epsilon_0 + j - h_1)(\epsilon_0 - j - h_1 - 1). \tag{4.4}$$

It follows that, at $j = 0$, the positivity of norm of all states requires either that $\epsilon_0 \geq h_1 + 1$ or that $\epsilon_0 = h_1$. This turns out to be the complete set of necessary and sufficient conditions for the existence of unitary representations. Representations with $j = 0$ and $\epsilon_0 = h_1 + 1$ or $\epsilon_0 = h_1$ are both short. The representation at $\epsilon_0 = h_1$ is an isolated short representation since there is no representation in the energy gap $h_1 \leq \epsilon_0 \leq (h_1 + 1)$; its first zero norm state occurs at level one. The first zero norm state in the $j = 0$ representation at $\epsilon_0 = h_1 + 1$ occurs at level 2 and is given by $|s_2\rangle$.

In summary, short representations occur when the highest weights of the primary state satisfy one of the following conditions [7].

$$\begin{aligned} \epsilon_0 =& j + h_1 + 1 \quad \text{when j} \geq 0, \\ \epsilon_0 =& h_1 \quad \text{when j} = 0. \end{aligned} \tag{4.5}$$

The last condition gives an isolated short representation.

### 4.2.2 Null Vectors and Character Decomposition of a Long Representation at the Unitarity Threshold

As we have explained in the previous subsection, short representations of the $d = 3$ superconformal algebra are of two sorts. The energy of a 'regular' short representation is given by $\epsilon_0 = j + h_1 + 1$. The null states of this representation transform in an

[5]When $j \neq 0$, the norm of $|s_2\rangle$ had to be proportional to $(\epsilon_0 - j - h_1 - 1)$ simply because the norm of $|s_2\rangle$ must vanish whenever $|Zn_1\rangle$ is of zero norm. The algebra that leads to this result is correct even at $j = 0$ (i.e. when $|Zn_1\rangle$ is ill defined).

irreducible representation of the algebra. When $j \neq 0$ the highest weights of the primary at the head of this null irreducible representation is given in terms of the highest weights of the representation itself by $\epsilon_0' = \epsilon_0 + \frac{1}{2}, \quad j' = j - \frac{1}{2}, \quad h_1' = h_1 + 1, \quad h_i' = h_i$. Note that $\epsilon_0' - j' - h_1' - 1 = \epsilon_0 - j - h_1 - 1 = 0$, so that the null states also transform in a regular short representation. As the union of the ordinary and null states of such a short representation is identical to the state content of a long representation at the edge of the unitarity bound, we conclude that

$$\lim_{\delta \to 0} \chi[j + h_1 + 1 + \delta, j, h_1, h_j] = \chi[j + h_1 + 1, j, h_1, h_j] + \chi[j + h_1 + 3/2, j - \frac{1}{2}, h_1 + 1, h_j] \tag{4.6}$$

where $\chi[\epsilon_0, j, h_i]$ denotes the supercharacter of the irreducible representation with energy $\epsilon_0$, $SO(3)$ highest weight $j$ and $SO(n)$ highest weights $\{h_i\}$. Note that the $\chi$ s appearing on the RHS of (4.6) are the supercharacters corresponding to short representations.

On the other hand when $j = 0$ the null states of the regular short representation occur at level 2 and are labelled by a primary with highest weights $\epsilon_0' = \epsilon_0 + 1, \quad j' = 0, h_1' = h_1 + 2, \quad h_i' = h_i$. Note in particular that $j' = 0$ and $\epsilon_0' - h_1' = \epsilon_0 - h_1 - 1 = 0$. It follows that the null states of this representation transform in an isolated short representation, and we conclude

$$\lim_{\delta \to 0} \chi[h_1 + 1 + \delta, j = 0, h_1, h_j] = \chi[h_1 + 1, j = 0, h_1, h_j] + \chi[h_1 + 2, j = 0, h_1 + 2, h_j] \tag{4.7}$$

Recall that isolated short representations are separated from all other representations with the same $SO(3)$ and $SO(n)$ quantum numbers by a gap in energy. As a consequence it is not possible to 'approach' such representations with long representations; consequently we have no equivalent of (4.7) at energies equal to $h_1 + \delta$.

For use below we define some notation. We will use $c(j,h_i)$ (with $i = 1,2,\ldots,[\frac{n}{2}]$) to denote a regular short representation with $SO(3)$ and $SO(n)$ highest weights $j$, $h_i$ respectively and $\epsilon_0 = j+h_1+1$ (when $j \geq 0$). We will also use the symbol $c(-\frac{1}{2},h_1,h_j)$ (with $h_1 \geq h_2 - 1$) to denote the isolated short representation with $SO(3)$ quantum number 0, $SO(n)$ quantum numbers $h_1+1, h_j$ (here $j = 2,3,\ldots,[\frac{n}{2}]$) respectively and $\epsilon_0 = h_1 + 1$. The utility of this notation will become apparent below.

### 4.2.3 Indices

The state content of any unitary superconformal quantum field theory may be decomposed into a sum of an (in general infinite number of) irreducible representations of the superconformal algebra. This state content is completely determined by specifying the number of times any given representation occurs in this decomposition. Consider any linear combination of the multiplicities of short representations. If this linear combination evaluates to zero on every collection of representations that appears on the RHS of each of (4.6) and (4.7) (for all values of parameters), it is said to be an Index. It follows from this definition that Indices are unaffected by all possible pairing up of short representations into long representations, and so are invariant under any deformation of superconformal Hilbert space under which the spectrum evolves continuously. We now proceed to list these Indices.

1. The simplest Indices are simply given by the multiplicities of representations in the spectrum that never appear on the RHS of (4.7) and (4.6) (for any values of the quantum numbers of the long representations on the LHS of those equations). All such representations are easy to list; they are $SO(3)$ singlet

isolated representations whose $SO(n)$ quantum number $h_1 - |h_2| \leq 1$ where $h_1$ and $h_2$ are both either integers or half integers, and $h_1 \geq |h_2| \geq 0$.

2. We can also construct Indices from linear combinations of the multiplicities of representations that do appear on the RHS of (4.7) and (4.6). The complete list of such linear combinations is given by

$$I_{M,\{h_j\}} = \sum_{p=-1}^{M-|h_2|} (-1)^{p+1} n\{c(\frac{p}{2}, M-p, h_j\}, \tag{4.8}$$

where $n[R]$ denotes the multiplicities of representations of type $R$ and the Index label $M$ is the value of $h_1+2j$ for every regular short representation that appears in the sum above. Thus $M \geq |h_2|$ and both $M$ and $h_2$ are either integers or half-integers. Also the set $\{h_j\}$ must satisfy the condition $h_2 \geq h_3 ..... \geq |h_{[\frac{n}{2}]}|$ where all the $h_i$ are either integers or all are half-integers.

### 4.2.4 Minimally BPS states: distinguished supercharge and commuting superalgebra

We will now describe states that are annihilated by at least one supercharge and its conjugate. Consider the special supercharge $Q$ with charges ($j = -\frac{1}{2}, h_1 = 1, h_i = 0, \epsilon_0 = \frac{1}{2}$). Let $S = Q^\dagger$; it is easily verified that

$$\{S, Q\} = \Delta = \epsilon_0 - (h_1 + j) \tag{4.9}$$

Below we will be interested in a partition function over states annihilated by $Q$. Clearly all such states transform in irreducible representations of that subalgebra of the superconformal algebra that commutes with $Q, S$ and hence $\Delta$. This subalgebra is easily determined to be a real form of the supergroup $D(\frac{n-2}{2}, 1)$ or $B(\frac{n-3}{2}, 1)$, depending on whether $n$ is even or odd. We follow the same notation as [7].

The bosonic subgroup of this commuting superalgebra is $SO(2,1) \times SO(n-2)$. The usual Cartan charge of $SO(2,1)$ (the $SO(2)$ rotation) and the Cartan charges of $SO(n-2)$ are given in terms of the Cartan elements of the parent superconformal algebra by

$$E = \epsilon_0 + j, \quad H_i = h_{i+1} \quad \left(\text{with i} = 1, 2, \ldots, [\frac{\text{n}-2}{2}]\right). \tag{4.10}$$

### 4.2.5 A Trace formula for the general Index and its Character Decomposition

Let us define the Witten Index

$$I^W = \mathrm{Tr}_R[(-1)^F \exp(-\beta\Delta + G)], \tag{4.11}$$

where the trace is evaluated over any Hilbert space $R$ that hosts a representation (not necessarily irreducible) of the superconformal algebra. Here $F$ is the Fermion number operator; by the spin statistics theorem $F = 2j$ in any quantum field theory. $G$ is any element of the subalgebra that commutes with $\{S, Q, \Delta\}$; by a similarity transformation, $G$ may be rotated into a linear combination of the Cartan generators of the subalgebra.

The Witten Index (4.11) receives contributions only from states that are annihilated by both $Q$ and $S$ (all other states yield contributions that cancel in pairs) and have $\Delta = 0$. So, it is independent of $\zeta$. The usual arguments[2] also ensure that $I^W$ is an Index; consequently it must be possible to expand $I^W$ as a linear sum over the Indices defined in the previous section. In fact it is easy to check that for any representation A(reducible or irreducible),

$$I^W(A) = \sum_{M,\{h_i\}} I_{M,\{h_i\}} \chi_{sub}(M+2, h_i) + \sum_{\{h_j\}, h_1 - |h_2| = 0,1} n\{c(-\frac{1}{2}, h_1 - 1, h_i)\} \chi_{sub}(h_1, h_i). \tag{4.12}$$

where $\chi_{sub}(E, H_i)$ (with $i = 1, 2, \ldots, [\frac{n-2}{2}]$) is the supercharacter of the subalgebra[6] with $E$ and $H_i$ being the highest weights of a representation of the subalgebra in the convention defined by (4.10). In the first term on the RHS of (4.12) the sum runs over all the values of $M, \{h_j\}$ for which $I_{M,\{h_j\}}$ is defined (see below (4.8)). In the second term the sum runs over all the values of the set $\{h_j\}$ such that $h_2 \geq h_3..... \geq |h_{[\frac{n}{2}]}|$. In order to obtain (4.13) we have used

$$I^W(c(j, h_1, h_j)) = (-1)^{2j+1} \chi_{sub}(2j + h_1 + 2, h_i) \tag{4.13}$$

$$I^W(c(j = -\frac{1}{2}, h_1, h_j)) = \chi_{sub}(h_1 + 1, h_j) \tag{4.14}$$

Equation (4.13) asserts that the set of $\Delta = 0$ states (the only states that contribute to the Witten Index) in any short irreducible representation of the superconformal algebra transform in a single irreducible representation of the commuting subalgebra. In the case of regular short representations, the primary of the full representation has $\Delta = 1$. The primary of the subalgebra is obtained by acting on the primary of the full representation with a supercharge with quantum numbers ($j = \frac{1}{2}, h_1 = 1, h_i = 0, \epsilon_0 = \frac{1}{2}, \Delta = -1$). On the other hand the highest weight of an isolated superconformal

[6]The supercharacter of a representation $R$ is defined as $\chi_{sub}(R) = \mathrm{tr}_R(-1)^F \mathrm{tr}\ e^{\mu \cdot \mathbf{H}}$, where $\mu \cdot \mathbf{H}$ is some linear combination of the Cartan generators specified by a chemical potential vector $\mu$. $F$ is defined to anticommute with $Q$ and commute with the bosonic part of the algebra. The highest weight state is *always taken* to have $F = 0$.

short primary itself has $\Delta = 0$, and so is also a primary of the commuting sub super algebra. Equation (4.12) follows immediately from these facts.

Note that every Index that appears in the list of subsection 2.3 appears as the coefficient of a distinct subalgebra supercharacter in (4.12). As supercharacters of distinct irreducible representations are linearly independent, it follows that knowledge of $I^W$ is sufficient to reconstruct all superconformal Indices of the algebra. In this sense (4.12) is the most general Index that is possible to construct from the superconformal algebra alone.

### 4.2.6 The Index over M theory multi gravitons in $AdS_4 \times S^7$

We will now compute the Witten Index defined above in specific examples of three dimensional superconformal field theories. In this subsection we focus on the world volume theory of the M2 brane in the large $N$ limit. The corresponding theory has supersymmetries and 16 superconformal symmetries. The bosonic subgroup of the relevant superconformal algebra is $SO(3, 2) \times SO(8)$. We take the supercharges to transform in the vector representation of $SO(8)$; this convention is related to the one used in much of literature on this theory by a triality flip.

In the strict large $N$ limit, the Index over the M2 brane conformal field theory is simply the Index over the Fock space of supergravitons for M theory on $AdS_4 \times S^7$ [1, 57]. In order to compute this quantity we first compute the Index over single graviton states; the Index over multi gravitons is given by the appropriate Bose-Fermi exponentiation (sometimes called the plethystic exponential).[7]

[7] The Index we will calculate is sensitive to $\frac{1}{16}$ BPS states. However, the $\frac{1}{8}$ BPS partition function has been calculated, even at finite $N$, in [58]

Single particle supergravitons in $AdS_4 \times S^7$ transform in an infinite class of representations of the superconformal algebra. The primaries for this spectrum have charges (see [59, 60]) ($\epsilon_0 = \frac{n}{2}$,$j = 0$,$h_1 = \frac{n}{2}$,$h_2 = \frac{n}{2}$, $h_3 = \frac{n}{2}$,$h_4 = -\frac{n}{2}$) ($h_1, h_2, h_3$ and $h_4$ denote $SO(8)$ highest weights in the orthogonal basis; recall $Q$s here are taken to transform in the vector rather than the spinor of $SO(8)$) where $n$ runs from 1 to $\infty$ (we are working with the $U(N)$ theory; $n = 1$ would be omitted for the $SU(N)$ theory).

It is not difficult to decompose each of these irreducible representations of the superconformal algebra into representations of the conformal algebra, and thereby compute the partition function and the Index over each of these representations. The necessary decompositions were performed in [59], and we have verified their results independently by means of the Racah Speiser algorithm. We direct the reader fo the Appendix of [31] for a description of this procedure.

The results are listed in Table 4.1 below.[8]

It is now a simple matter to compute the Index over single gravitons. The Witten Index for the $n^{th}$ graviton representation ($R_n$) is given by

$$\begin{aligned} I^W_{R_n} &= \mathrm{Tr}_{\Delta=0}\Big[(-1)^F x^{\epsilon_0+j} y_1^{H_1} y_2^{H_2} y_3^{H_3}\Big] \\ &= \sum_q \frac{(-1)^{2j_q} x^{(\epsilon_0+j)_q} \chi_q^{SO(6)}(y_1, y_2, y_3)}{1-x^2}, \end{aligned} \tag{4.15}$$

where $q$ runs over all conformal representations with $\Delta = 0$ that appear in the decomposition of $R_n$ in table 4.1. $H_1, H_2, H_3$ are the Cartan charges of $SO(6)$ in the 'orthogonal' basis that we always use in this chapter. $\chi^{SO(6)}$, the $SO(6)$ character, may

[8]Some of the conformal representations obtained in this decomposition are short (as conformal representations) when $n$ is either 1 or 2; the negative contributions in table 1 represent the charges of the null states, which physically are operators set to zero by the equations of motion. See [6]

Table 4.1: d=3 graviton spectrum

| range of $n$ | $\epsilon_0[SO(2)]$ | $SO(3)$ | $SO(8)$[orth.(Qs in vector)] | $\Delta$ | contribution |
|---|---|---|---|---|---|
| $n \geq 1$ | $\frac{n}{2}$ | $0$ | $(\frac{n}{2}, \frac{n}{2}, \frac{n}{2}, \frac{-n}{2})$ | $0$ | $+$ |
| $n \geq 1$ | $\frac{n+1}{2}$ | $\frac{1}{2}$ | $(\frac{n}{2}, \frac{n}{2}, \frac{n}{2}, \frac{-(n-2)}{2})$ | $0$ | $+$ |
| $n \geq 2$ | $\frac{n+2}{2}$ | $1$ | $(\frac{n}{2}, \frac{n}{2}, \frac{(n-2)}{2}, \frac{-(n-2)}{2})$ | $0$ | $+$ |
| $n \geq 2$ | $\frac{n+3}{2}$ | $\frac{3}{2}$ | $(\frac{n}{2}, \frac{(n-2)}{2}, \frac{(n-2)}{2}, \frac{-(n-2)}{2})$ | $0$ | $+$ |
| $n \geq 2$ | $\frac{n+4}{2}$ | $2$ | $(\frac{(n-2)}{2}, \frac{(n-2)}{2}, \frac{(n-2)}{2}, \frac{-(n-2)}{2})$ | $1$ | $+$ |
| $n \geq 2$ | $\frac{n+2}{2}$ | $0$ | $(\frac{n}{2}, \frac{n}{2}, \frac{n}{2}, \frac{-(n-4)}{2})$ | $1$ | $+$ |
| $n \geq 3$ | $\frac{n+3}{2}$ | $\frac{1}{2}$ | $(\frac{n}{2}, \frac{n}{2}, \frac{(n-2)}{2}, \frac{-(n-4)}{2})$ | $1$ | $+$ |
| $n \geq 3$ | $\frac{n+4}{2}$ | $1$ | $(\frac{n}{2}, \frac{(n-2)}{2}, \frac{(n-2)}{2}, \frac{-(n-4)}{2})$ | $1$ | $+$ |
| $n \geq 3$ | $\frac{n+5}{2}$ | $\frac{3}{2}$ | $(\frac{(n-2)}{2}, \frac{(n-2)}{2}, \frac{(n-2)}{2}, \frac{-(n-4)}{2})$ | $2$ | $+$ |
| $n \geq 4$ | $\frac{n+5}{2}$ | $\frac{1}{2}$ | $(\frac{n}{2}, \frac{(n-2)}{2}, \frac{(n-4)}{2}, \frac{-(n-4)}{2})$ | $2$ | $+$ |
| $n \geq 4$ | $\frac{n+7}{2}$ | $\frac{1}{2}$ | $(\frac{(n-2)}{2}, \frac{(n-4)}{2}, \frac{(n-4)}{2}, \frac{-(n-4)}{2})$ | $4$ | $+$ |
| $n \geq 4$ | $\frac{n+6}{2}$ | $1$ | $(\frac{(n-2)}{2}, \frac{(n-2)}{2}, \frac{(n-4)}{2}, \frac{-(n-4)}{2})$ | $3$ | $+$ |
| $n \geq 4$ | $\frac{n+4}{2}$ | $0$ | $(\frac{n}{2}, \frac{n}{2}, \frac{(n-4)}{2}, \frac{-(n-4)}{2})$ | $2$ | $+$ |
| $n \geq 4$ | $\frac{n+6}{2}$ | $0$ | $(\frac{n}{2}, \frac{(n-4)}{2}, \frac{(n-4)}{2}, \frac{-(n-4)}{2})$ | $3$ | $+$ |
| $n \geq 4$ | $\frac{n+8}{2}$ | $0$ | $(\frac{(n-4)}{2}, \frac{(n-4)}{2}, \frac{(n-4)}{2}, \frac{-(n-4)}{2})$ | $6$ | $+$ |
| $n = 1$ | $2$ | $\frac{1}{2}$ | $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ | $1$ | $-$ |
| $n = 1$ | $\frac{5}{2}$ | $0$ | $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ | $2$ | $-$ |
| $n = 2$ | $3$ | $0$ | $(1, 1, 0, 0)$ | $2$ | $-$ |
| $n = 2$ | $\frac{7}{2}$ | $\frac{1}{2}$ | $(1, 0, 0, 0)$ | $2$ | $-$ |
| $n = 2$ | $4$ | $1$ | $(0, 0, 0, 0)$ | $3$ | $-$ |

be computed using the Weyl character formula. The full Index over single gravitons is

$$I_{sp} = \sum_{n=2}^{\infty} I^W_{R_n} + I^W_{R_1}. \tag{4.16}$$

After some algebra we find

$$I_{sp} = \Big[ -x\left(x^2-1\right) y_1 y_2 y_3^2 + \sqrt{x}\sqrt{y_1}\sqrt{y_2}\left(x^3 - y_2 + y_1\left(x^3 y_2 - 1\right)\right) y_3^{3/2} - x\left(x^2-1\right)\left(y_1+y_2\right)\left(y_1 y_2 + 1\right) y_3 + \sqrt{x}\sqrt{y_1}\sqrt{y_2}\left(y_2 x^3 + y_1\left(x^3 - y_2\right) - 1\right) \sqrt{y_3} - x\left(x^2-1\right) y_1 y_2 \Big] \Big/ \Big[ \left(x^2-1\right)\left(\sqrt{x}\sqrt{y_1}\sqrt{y_2} - \sqrt{y_3}\right) \left(\sqrt{x}\sqrt{y_1}\sqrt{y_3} - \sqrt{y_2}\right)\left(\sqrt{x}\sqrt{y_2}\sqrt{y_3} - \sqrt{y_1}\right)\left(\sqrt{x} - \sqrt{y_1}\sqrt{y_2}\sqrt{y_3}\right) \Big]. \tag{4.17}$$

The Index over the Fock-space of gravitons may now be obtained from the above single particle Index using

$$I_{fock} = \exp\left(\sum_n \frac{1}{n} I_{sp}(x^n, y_1^n, y_2^n, y_3^n)\right). \tag{4.18}$$

In order to get a feel for this result, let us set $y_i = 1$. The single graviton Index reduces to

$$I_{sp} = \frac{2\sqrt{x}\left(2x + \sqrt{x} + 2\right)}{\left(\sqrt{x}-1\right)^2 (x+1)}. \tag{4.19}$$

In the high energy limit, $x \equiv e^{-\beta} \to 1$, this expression simplifies to $I_{sp} \approx \frac{20}{\beta^2}$ In this limit the expression for the full Witten Index $I_{fock}$ in (4.18) reduces to,

$$I_{fock} \approx \exp \frac{20\zeta(3)}{\beta^2} \tag{4.20}$$

It follows that the thermodynamic expectation value of $\epsilon_0 + j$ (which we denote by $E_{\rm av}^{\rm ind}$) is given by

$$E_{\rm av}^{\rm ind} = -\frac{\partial \ln I_{fock}}{\partial \beta} = \frac{40\zeta(3)}{\beta^3}. \tag{4.21}$$

The Index 'entropy' defined by

$$I_{fock} = \int dy \exp\{(-\beta y) + S_{\rm ind}(y)\}, \tag{4.22}$$

evaluates to

$$S_{\text{ind}}(E) = \frac{60\zeta(3)}{(40\zeta(3))^{\frac{2}{3}}} E^{\frac{2}{3}}. \tag{4.23}$$

It is instructive to compare this result with the relation between entropy and $E$ computed from the supersymmetric partition function, obtained by summing over all supersymmetric states with no $(-1)^F$ – once again in the gravity approximation. The single particle partition function evaluated on the $\Delta = 0$ states with all the other chemical potentials except the one corresponding to $E = \epsilon_0 + j$ set to zero is given by,

$$Z_{sp}(x) = \text{tr}_{\Delta=0} x^E = \frac{2\sqrt{x}(x+1)\left(x^{5/2} - 2x^2 + 2x^{3/2} + 2x - 3\sqrt{x} + 2\right)}{\left(\sqrt{x}-1\right)^4 (x^2-1)}, \tag{4.24}$$

where once again $x \equiv e^{-\beta}$, with $\beta$ being the chemical potential corresponding to $E = \epsilon_0 + j$. The bosonic and fermionic contributions to the partition function in (4.24) are respectively given by,

$$Z_{sp}^{\text{bose}}(x) = \text{tr}_{\Delta=0 \text{ bosons}} x^E = \frac{-\left(-x^4 + 4x^{7/2} - 6x^3 + x^2 - 4x^{3/2} + 6x - 4\sqrt{x}\right)}{(1-\sqrt{x})^5 (\sqrt{x}+1)(x+1)} \tag{4.25}$$

$$Z_{sp}^{fermi}(x) = \text{tr}_{\Delta=0 \text{ fermions}} x^E = \frac{-\left(-x^4 + x^2 - 4x^{3/2}\right)}{(1-\sqrt{x})^5 (\sqrt{x}+1)(x+1)} \tag{4.26}$$

To obtain the Index on the Fock space, we need to multi-particle the partition function above with the correct Bose-Fermi statistics. This leads to

$$Z_{fock} = \exp \sum_n \frac{Z_{sp}^{bose}(x^n) + (-1)^{n+1} Z_{sp}^{fermi}(x^n)}{n}. \tag{4.27}$$

We find, that for $\beta << 1$

$$\ln Z_{fock} = \frac{63\zeta(6)}{\beta^5}, \tag{4.28}$$

and a calculation similar to the one done above yields

$$S(E) = \frac{378\zeta(6)}{(315\zeta(6))^{\frac{5}{6}}} E^{\frac{5}{6}}. \tag{4.29}$$

which is the growth of states with energy of a six dimensional gas, an answer that could have been predicted on qualitative grounds. Recall that the theory of the worldvolume of the $M2$ brane has 4 supersymmetric transverse fluctuations and one supersymmetric derivative. Bosonic supersymmetric gravitons are in one to one correspondence with 'words' formed by acting on symmetric combinations of these scalars with an arbitrary number of derivatives. Consequently, supersymmetric gravitons are labelled by 5 integers $n_i$, $n_d$ (the number of occurrences of each of these four scalars $i = 1 \dots 4$ and the derivative $n_d$) and the energy of these gravitons is $E = \frac{1}{2}(\sum_i n_i) + n_d$. This is the same as the formula for the energy of massless photons in a five spatial dimensional rectangular box, four of whose sides are of length two and whose remaining side is of unit length, explaining the effective six dimensional growth.

We conclude that the growth of states in the effective Index entropy is slower than the growth of supersymmetric states in the system; this is a consequence of partial Bose-Fermi cancellations (due to the $(-1)^F$).

### 4.2.7 The Index on the worldvolume theory of a single $M2$ brane

We will now compute our Index over the worldvolume theory of a single $M2$ brane. For this free theory, the single particle state content is just the representation corresponding to $n = 1$ in Table 4.1 of the previous subsection. This means that it corresponds to the representation of the $d = 3$ superconformal group with the primary having charges $\epsilon_0 = \frac{1}{2}, j = 0$ and $SO(8)$ highest weights (in the convention described above) $[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}]$.

For the reader's convenience, we reproduce the conformal multiplets that appear in this representation in the Table below. Physically, these multiplets correspond to the 8 transverse scalars, their fermionic superpartners and the equations of motion for each of these fields.[9]

| letter | $\epsilon_0$ | $j$ | $[h_1, h_2, h_3, h_4]$ | $\Delta = \epsilon_0 - j - h_1$ |
|---|---|---|---|---|
| $\phi^a$ | $\frac{1}{2}$ | 0 | $[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}]$ | 0 |
| $\psi^a$ | 1 | $\frac{1}{2}$ | $[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$ | 0 |
| $\partial\!\!\!/\psi^a = 0$ | 2 | $\frac{1}{2}$ | $[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$ | 1 |
| $\partial^2\phi^a = 0$ | $\frac{5}{2}$ | 0 | $[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}]$ | 2 |

(4.30)

The Index over these states is

$$\begin{aligned} I^{\mathrm{sp}}_{M_2}(x, y_i) &= \mathrm{Tr}\left[(-1)^F x^{\epsilon_0+j} y_1^{H_1} y_2^{H_2} y_3^{H_3}\right] \\ &= \frac{x^{\frac{1}{2}}\left(1 + y_1 y_2 + y_1 y_3 + y_2 y_3\right) - x^{\frac{3}{2}}\left(y_1 + y_2 + y_3 + y_1 y_2 y_3\right)}{\left(y_1 y_2 y_3\right)^{\frac{1}{2}}\left(1 - x^2\right)} \end{aligned} \tag{4.31}$$

For simplicity, let us set $y_i \to 1$. Then, we find

$$I^{\mathrm{sp}}_{M_2}(x, y_i = 1) = \frac{4x^{\frac{1}{2}}}{1 + x} \tag{4.32}$$

Multiparticling this Index, to get the Index over the Fock space on the $M_2$ brane, we

[9]Please see [61, 62] and references therein for more details on this worldvolume theory and [63] for some recent work.

find that

$$\begin{aligned} I_{M_2}(x, y_i = 1) &= \exp \sum_{n \geq 1} \frac{I_{M_2}(x^n, y_i = 1)}{n} \\ &= \left( \prod_{n \geq 0} \frac{1 - x^{2n+\frac{3}{2}}}{1 - x^{2n+\frac{1}{2}}} \right)^4 \end{aligned} \tag{4.33}$$

At high temperatures $x \equiv e^{-\beta} \to 1$, the Index grows as

$$I_{M_2}|_{x \to 1, y_i = 1} = \left( \frac{\beta}{2} \right)^4 \tag{4.34}$$

The single particle supersymmetric partition function, obtained by summing over all $\Delta = 0$ single particle states with no $(-1)^F$ is,

$$\begin{aligned} Z^{\text{susy,sp}}_{M_2}(x, y_i) &= \mathrm{Tr}_{\Delta=0} \left[ x^{\epsilon_0 + j} y_1^{H_1} y_2^{H_2} y_3^{H_3} \right] \\ &= \frac{x^{\frac{1}{2}} (1 + y_1 y_2 + y_1 y_3 + y_2 y_3) + x^{\frac{3}{2}} (y_1 + y_2 + y_3 + y_1 y_2 y_3)}{(y_1 y_2 y_3)^{\frac{1}{2}} (1 - x^2)} \end{aligned} \tag{4.35}$$

Setting $y_i \to 1$,

$$Z^{\text{susy,sp}}_{M_2}(x, y_i = 1) = \frac{4x^{\frac{1}{2}}}{1 - x} \tag{4.36}$$

with individual contributions from bosons and fermions being

$$\begin{aligned} Z^{\text{susy,sp,bose}}_{M_2}(x) &= \mathrm{tr}_{\Delta=0 \ \text{bosons}} x^E = \frac{4x^{\frac{1}{2}}}{(1 - x^2)} \\ Z^{\text{susy,sp,fermi}}_{M_2}(x) &= \mathrm{tr}_{\Delta=0 \ \text{fermions}} x^E = \frac{4x^{\frac{3}{2}}}{(1 - x^2)} \end{aligned} \tag{4.37}$$

Finally, multi-particling this partition function with the appropriate bose-fermi statistics, we find that

$$Z_{M_2}(x, y_i = 1) = \left( \prod_{n \geq 0} \frac{1 + x^{2n+\frac{3}{2}}}{1 - x^{2n+\frac{1}{2}}} \right)^4 \tag{4.38}$$

At high temperatures $x \to 1$, the supersymmetric partition function grows as

$$Z_{M_2}(x \to 1, y_i = 1) \approx \exp \left\{ \frac{\pi^2}{2\beta} \right\} \tag{4.39}$$

Note, that this partition function grows significantly faster at high temperatures than the Index (4.33).

### 4.2.8 Index over Chern Simons Matter Theories

In this subsection, we will calculate the Witten Index described above for a class of the superconformal Chern Simons matter theories recently studied by Gaiotto and Yin [55]. The theories studied by these authors are three dimensional Chern Simons gauge theories coupled to matter fields; we will focus on examples that enjoy invariance under a superalgebra consisting of 4 Qs and 4 Ss (i.e. the $R$ symmetry of these theories is $SO(2)$). The matter fields, which may be thought of as dimensionally reduced $d = 4$ chiral multiplets, carry the only propagating degrees of freedom. The general constructions of Gaiotto and Yin allow the possibility of nonzero superpotentials with a coupling $\alpha$ that flows in the infra-red to a fixed point of order $\frac{1}{k}$ where $k$ is the level of the Chern Simons theory. In our analysis below we will focus on the limit of large $k$. In this limit, the theory is 'free' and moreover we may treat $\frac{1}{k}$ as a continuous parameter. The arguments above then indicate Index that we compute below for the free theory will be invariant under small deformations of $\frac{1}{k}$.

Consider this free conformal 3 dimensional theory on $S^2$. We are interested in calculating the letter partition function (i.e. the single particle partition function) for the propagating fields which comprise a complex scalar $\phi$ and its fermionic superpartner $\psi$. This may be done by enumerating all operators, linear in these fields, modulo those operators that are set to zero by the equations of motion. We will be interested in keeping track of several charges: the energy $\epsilon_0$, $SO(3)$ angular momentum $j$, $SO(2)$ R-charge $h$ and $\Delta = \epsilon_0 - h - j$ of our states. The following table (which lists these charges) is useful for that purpose

| letter | $\epsilon_0$ | $j$ | $h$ | $\Delta = \epsilon_0 - j - h$ |
|---|---|---|---|---|
| $\phi$ | $\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $0$ |
| $\phi*$ | $\frac{1}{2}$ | $0$ | $\frac{-1}{2}$ | $1$ |
| $\psi$ | $1$ | $\frac{1}{2}$ | $\frac{-1}{2}$ | $1$ |
| $\psi*$ | $1$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $0$ |
| $\partial_\mu$ | $1$ | $\{\pm 1, 0\}$ | $0$ | $\{0, 2, 1\}$ |
| $\partial_\mu \sigma^\mu \psi = 0$ | $2$ | $\frac{1}{2}$ | $\frac{-1}{2}$ | $2$ |
| $\partial_\mu \sigma^\mu \psi^* = 0$ | $2$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $1$ |
| $\partial^2 \phi = 0$ | $\frac{5}{2}$ | $0$ | $\frac{1}{2}$ | $2$ |
| $\partial^2 \phi^* = 0$ | $\frac{5}{2}$ | $0$ | $\frac{-1}{2}$ | $3$ |

(4.40)

The last four lines, with equations of motion count with minus signs in the partition function. The list above comprises two separate irreducible representations of the superconformal algebra. $\phi$, $\psi$ and derivatives on these letters make up one representation. The other representation consists of the conjugate fields.

Let the partition functions over these two representations be denoted by $z_1$ and $z_2$. We find

$$\begin{aligned} z_1[x, y, t] &= \mathrm{tr}_{\phi,\psi,\ldots}(x^{2\epsilon_0} y^{2j} t^h) = \frac{t^{\frac{1}{2}} x(1+x^2) + t^{\frac{-1}{2}} x^2 (y + 1/y)}{(1 - x^2 y^2)(1 - x^2/y^2)} \\ z_2[x, y, t] &= \mathrm{tr}_{\phi^*,\psi^*,\ldots}(x^{2\epsilon_0} y^{2j} t^{2h}) = \frac{t^{\frac{-1}{2}} x(1+x^2) + t^{\frac{1}{2}} x^2 (y + 1/y)}{(1 - x^2 y^2)(1 - x^2/y^2)} \end{aligned} \tag{4.41}$$

The Index (4.11) over single particle states is obtained by setting $t \to 1/x, y \to -1$

$$\begin{aligned}
I_1[x] &= z_1[x,-1,1/x] = \mathrm{tr}((-1)^F(x)^{2\epsilon_0-h}) = \frac{x^{\frac{1}{2}}}{1-x^2} \\
I_2[x] &= z_2[x,-1,1/x] = \mathrm{tr}((-1)^F x^{2\epsilon_0-h}) = \frac{-x^{\frac{3}{2}}}{1-x^2} \\
I[x] &= I_1[x] + I_2[x] = \frac{x^{\frac{1}{2}}}{1+x}
\end{aligned} \tag{4.42}$$

In terms of these quantities, the Index of the full theory is given by[3, 4]

$$I^W = \int DU \exp\left[\sum_{n=1}^{\infty}\sum_m \frac{I(x^n)}{n} Tr_{R_m}(U^n)\right] \tag{4.43}$$

where $m$ run over the chiral multiplets of the theory, which are taken to transform in the $R_m$ representation of $U(N)$, and $Tr_{R_m}$ is the trace of the group element in the $R_m^{th}$ representation of $U(N)$.

In the large $N$ limit the integral over $U$ in (4.43) may be converted into an integral over the eigenvalue distribution of $U$, $\rho(\theta)$, which, in turn, may be computed via saddle points.[10] The Fourier coefficients of this eigenvalue density function are given by:

$$\rho_n = \int_{-\pi}^{\pi} \rho(\theta)\cos(n\theta) \tag{4.44}$$

### Adjoint Matter

In order to get a feel for this formula, we specialize to a particular choice of matter field content. We consider a theory with $c$ matter fields all in the adjoint representation. In the large $N$ limit the Index is given by

$$\begin{aligned}
\mathcal{I}(x) &= Tr_{\text{coloursinglets}}(-1)^F x^{2\epsilon_0-h} \\
&= \int d\rho_n \exp\left(-N^2\sum_{n=1}^{\infty}\frac{1}{n}(1-cI[x^n])\rho_n^2\right)
\end{aligned} \tag{4.45}$$

[10]Note that $N\rho(\theta)d\theta$ gives the number of eigenvalues between $e^{i\theta}$ and $e^{i(\theta+d\theta)}$ and $\int_{-\pi}^{\pi}\rho(\theta)d\theta = 1, \quad \rho(\theta) \geq 0$

The behaviour of this Index as a function of $x$ is dramatically different for $c \leq 2$ and $c \geq 3$. In order to see this note that at any given value of $x$, the saddle point occurs at $\rho(\theta) = \frac{1}{2\pi}$ i.e $\rho_0 = 1, \rho_n = 0, n > 0$ provided that[3, 4]

$$1 - cI[x^n] > 0, \forall n \tag{4.46}$$

In this case the saddle point contribution to the Index vanishes; the leading contribution to the integral is then from the Gaussian fluctuations about this saddle point. Under these conditions the logarithm of the Index or the 'free-energy' a [11] is then of order 1 in the $\frac{1}{N}$ expansion.

It is easy to check that (4.46) is satisfied at all values of $x$ (which must lie between zero and one in order for (4.11) to be well defined) when $c \leq 2$. On the other hand, if $c \geq 3$ this condition is only met for

$$x < \left( \frac{1}{2} \left( c - \sqrt{c^2 - 4} \right) \right)^2 \tag{4.47}$$

At this value of $x$ the coefficient of $\rho_1^2$ in (4.45) switches sign and the saddle point above with a uniform eigenvalue distribution is no longer valid. The new saddle point that dominates this integral above this value of $x$, has a Gross-Witten type gap in the eigenvalue distribution. The Index undergoes a large $N$ first order phase transition at the critical temperature listed in (4.47). At and above this temperature the 'free-energy' is of order $N^2$.

Note that $I(1) = \frac{1}{2}$. It follows that the Index is well defined even at strictly infinite temperature This is unlike the logarithm of the actual partition function of

[11] We use this term somewhat loosely, since we are referring here to an Index and not a partition function

the same theory, whose $x \to 1$ limit scales like $N^2/(1-x)^2$ as $x \to 1$ (for all values of $c$) reflecting the $T^2$ dependence of a 2+1 dimensional field theory. This difference between the high temperature limits of the Index and the partition function reflects the large cancellations of supersymmetric states in their contribution to the Index.

**Fundamental Matter**

As another special example, let us consider a theory whose $N_f$ matter fields all transform in the fundamental representation of $U(N)$. We take the Veneziano limit: $N_c \to \infty, c = \frac{N_f}{N_c}$ fixed. The Index for the theory is now given by

$$\begin{aligned}\mathcal{I}(x) &= Tr_{\text{coloursinglets}}(-1)^F x^{2\epsilon_0 - h} \\ &= \int d\rho_n \exp(-N^2 \sum_{n=1}^{\infty} \frac{(\rho_n - cI[x^n])^2 - c^2 I[x^n]^2}{n})\end{aligned} \tag{4.48}$$

At low temperatures the integral in (4.48) is dominated by the saddle point

$$\rho_n = cI(x^n). \tag{4.49}$$

As the temperature is raised the integral in (4.48) undergoes a Gross-Witten type phase transition when $c$ is large enough. This is easiest to appreciate in the limit $c \gg 1$. In this limit $\rho_1 = \frac{1}{2}$ in the low temperature phase when at $x \approx \frac{1}{4c^2}$, and $\rho_n = \frac{1}{2^n c^{n-1}} \ll 1$. At approximately this value of $x$ the low temperature eigenvalue distribution $\rho(\theta)$ formally turns negative at $\theta = \pi$. This is physically unacceptable (as an eigenvalue density is, by definition, intrinsically positive). In actual fact the system undergoes a phase transition at this value of $x$. At large $c$ this phase transition is very similar to the one described by Gross and Witen in [64] and in a more closely related context by [65]. The high temperature eigenvalue distribution is 'gapped' i.e. it has support on only a subset (centered about zero) of the interval $(-\pi, \pi)$.

For this phase transition to occur, we need $c \geq 3$. To arrive at this result, we notice that the distribution (3.87) implies

$$\lim_{x \to 1-} \rho(\pi) = \lim_{x \to 1-} \rho(-\pi) = \frac{1}{\pi}\left(\frac{1}{2} - \frac{c}{4}\right) \tag{4.50}$$

So, for $c \geq 3$, $\rho(\pi)$ would always turn negative for some value of $x$. Beyond this temperature the saddle point (3.87) is no longer valid.

## 4.3 d=6

### 4.3.1 The Superconformal Algebra and its Unitary Representations

The bosonic subalgebra of the $d = 6$ superconformal algebra is $SO(6,2) \otimes Sp(2n)$ (the conformal algebra times the R symmetry algebra). The anticommuting generators in this algebra may be divided into the generators of supersymmetry ($Q$) and the generators of superconformal symmetries ($S$). Supersymmetry generators transform in the fundamental representation of the R-symmetry group $Sp(2n)$, [12] have charge half under dilatations (the $SO(2)$ factor of the compact $SO(6) \otimes SO(2) \in SO(6,2)$) and are chiral spinors under the $SO(6)$ factor of the same decomposition. Superconformal generators $S_i^\mu = (Q_\mu^i)^\dagger$ transform in the anti-chiral spinor representation of $SO(6)$, have scaling dimension (dilatation charge) $(-\frac{1}{2})$, and also transform in the anti-fundamental representation of the R-symmetry group. The charges of these generators are given in more detail in the appendix of [31]. In our notation for su-

[12] With our conventions, $Sp(2n)$ is of rank $n$. $Sp(2) = SO(3)$ and $Sp(4) = SO(5)$.

persymmetry generators $i$ is an $SO(6)$ spinor Index while $\mu$ is an $R$ symmetry vector Index.

The commutation relations for this superalgebra are described in detail in [7]. As usual, the anticommutator between two supersymmetries is proportional to momentum times an $R$ symmetry delta function, and the anticommutator between two superconformal generators is obtained by taking the Hermitian conjugate of these relations. The most interesting relationship in the algebra is the anticommutator between $Q$ and $S$. Schematically

$$\{S_i^\mu, Q_\nu^j\} \sim \delta_\nu^\mu T_i^j - \delta_i^j M_\nu^\mu$$

Here $T^{ij}$ are the $U(4) \sim SO(6)\times SO(2)$ generators and $M_{\mu\nu}$ are the $Sp(2n)$ generators. The energy $\epsilon_0$, $SO(6)$ highest weight (denoted by $h_1, h_2$ and $h_3$ in the orthogonal basis [13]) and the R-symmetry highest weights $(k, k_1 \ldots, k_{(n-1)})$ of primary states form a complete set of labels for the representation in question. We use a non-standard normalization for the R-symmetry weights. In particular,

$$k = \frac{k^o}{2}, \quad k_i = \frac{k_i^o}{2} \tag{4.51}$$

Here $[k^o, k_i^o]$ are the highest weights of $Sp(2n)$ in the orthogonal basis.[14] As we have noted above, at the level of the algebra, $SO(2) \times SO(6) \sim U(4)$. We will sometimes find it convenient to label primaries by eigenvalues $c_i$ under the generators $T_i^i \equiv T_i$ of $U(4)$[15] rather than by the energy and $SO(6)$ weights. For any highest

[13] $h_i$ are eigenvalues under rotations in orthogonal 2 planes in $R^n$. Thus, for instance, $\{h_i\} = (1, 0, 0)$ in the vector representation. They are either integer or half integer and satisfy the constraint $h_1 \geq h_2 \geq |h_3| \geq 0$

[14] In the orthogonal basis, the Cartans of $Sp(2n)$ are $2n \times 2n$ matrices with elements $\mathrm{diag}(i\sigma_2, 0, 0 \ldots), \mathrm{diag}(0, i\sigma_2, 0, 0, \ldots), \ldots$, where each 0 is shorthand for a $2 \times 2$ matrix

[15] In the defining representation of $U(4)$ $(T_i)_b^a = \delta_i^a \delta_b^i$.

weight $(c_1, c_2, c_3, c_4)$ the eigenvalues satisfy $c_1 \geq c_2 \geq c_3 \geq c_4 \geq 0$ and $c_i$ s are always integers. For future reference we note the change of basis between the Cartan elements $\epsilon_0, h_1, h_2, h_3$ (the energy and 3 orthogonal $SO(6)$ Cartan generators) and $T_1, T_2, T_3, T_4$:

$$\begin{aligned}
\epsilon_0 &= \frac{1}{2}(T_1 + T_2 + T_3 + T_4) \\
h_1 &= \frac{1}{2}(T_1 + T_2 - T_3 - T_4) \\
h_2 &= \frac{1}{2}(T_1 - T_2 + T_3 - T_4) \\
h_3 &= \frac{1}{2}(T_1 - T_2 - T_3 + T_4)
\end{aligned} \tag{4.52}$$

As in the case of the $d = 3$ algebra, any irreducible representation of the superconformal algebra may be decomposed into a finite number of distinct irreducible representations of the conformal algebra. The latter are labeled by their own conformal primary states, which have a definite lowest energy and transform in a given irreducible representation of $SO(6)$.

We will now analyse the constraints imposed by unitarity on the quantum numbers of primary states; for this purpose we will find it convenient to use the $U(4)$ labeling of primaries introduced above. Let $Q^i_\mu$ $i = 1, \cdots, 4.$ and $\mu = \pm 1, \cdots, \pm n$ denote the supersymmetry whose charge under $U(4)$ Cartan $T_j$ are $\delta^i_j$ and under the R-symmetry Cartan $M_\nu$ is (sign of$\mu$) $\times\ \delta^\nu_{|\mu|}$. The superconformal generators are $S^\mu_i = (Q^i_\mu)^\dagger$ and therefore they have the same charges as $Q^i_\mu$ but with opposite sign.

### 4.3.2 Norms and Null States

In this subsection we study unitarity restrictions (and the resultant structure of null states) of representations of the superconformal algebra. This analysis turns out to be a little more intricate than its $d = 3$ counterpart.

As we have seen above, states in the same representation of the superconformal algebra do not all have the same norm. However states that lie within the same representation of the maximal compact subgroup of the algebra, $U(4) \times Sp(2n)$, do have the same norm. Consequently, in order to examine the constraints from unitarity, we need only examine one state per representation of this compact subalgebra.

In order to study the restrictions imposed by unitarity at level $\ell$ we should, in principle, study all states obtained by acting with the tensor product of an arbitrary combination of $\ell$ supersymmetries on the set of primary states of an irreducible representation of the superconformal algebra. This set of states may be Clebsh Gordan decomposed into a sum of irreducible representations of $U(4) \times Sp(2n)$; and we should compute the norm of at least one state in each of these representations, and ensure its positivity in order to guarantee unitarity. However this problem is significantly simplified by the observation that the most stringent condition on unitarity occurs in those states that transform in the 'largest' $Sp(2n)$ [66]. Now it is easy to construct a state in the largest $Sp(2n)$ representation: one simply acts on those primary states that are $Sp(2n)$ highest weight with $\ell$ $Sp(2n)$ highest weight supersymmetries, i.e. supersymmetries of the form $Q^i_1$. This prescription completely fixes the $Sp(2n)$ quantum numbers of the states we will study in this section. All that remains is to study the decomposition of all such states into irreducible representations of $U(4)$ and to compute the norm of one state in each of these representations.

The decomposition of the states of interest into $U(4)$ representations at level $\ell$ is easily performed using Young Tableaux techniques. The set of U(4) tableaux for representations of the descendants is obtained by adding $\ell$ boxes to the tableaux of the primary in all possible ways that give rise to a legal tableaux, subject to the restriction that no two 'new' boxes occur on the same row (this restriction is forced on us by the antisymmetry of the $Q^i_1$ operators). Note, that in this decomposition, no representation occurs more than once.[16]

It is not too difficult to find an explicit formula for the highest weight states of each of these representations. Let us define the operators $\left(A^i = \sum_{j=1}^{i} Q^j_1 \Upsilon^i_j\right) \quad i = 1, \cdots, 4$ where $\Upsilon^i_j$ are functions of the $U(4)$ generators defined by

$$
\begin{aligned}
\Upsilon^j_j =& \text{Identity} \quad \text{(no sum over j)} \\
\Upsilon^4_1 =& -\left[T^2_1 T^3_2 T^4_3 \left(\frac{(T_3 - T_4 + 1)(T_2 - T_4 + 2)}{(T_3 - T_4)(T_2 - T_4 + 1)}\right)\right. \\
&\left. - T^3_2 T^4_3 T^2_1 \left(\frac{T_3 - T_4 + 1}{T_3 - T_4}\right) - T^4_3 T^2_1 T^3_2 \left(\frac{T_2 - T_4 + 2}{T_2 - T_4 + 1}\right) + T^4_3 T^3_2 T^2_1\right] \left(\frac{1}{T_1 - T_4 + 2}\right) \\
\Upsilon^4_2 =& -\left(T^4_3 T^3_2 - T^3_2 T^4_3 \left(\frac{T_3 - T_4 + 1}{T_3 - T_4}\right)\right) \left(\frac{1}{T_2 - T_4 + 1}\right) \\
\Upsilon^4_3 =& -T^4_3 \left(\frac{1}{T_3 - T_4}\right) \\
\Upsilon^3_1 =& -\left(T^3_2 T^2_1 - T^2_1 T^3_2 \left(\frac{T_2 - T_3 + 1}{T_2 - T_3}\right)\right) \left(\frac{1}{T_1 - T_3 + 1}\right) \\
\Upsilon^3_2 =& -T^3_2 \left(\frac{1}{T_2 - T_3}\right) \\
\Upsilon^2_1 =& -T^2_1 \left(\frac{1}{T_1 - T_2}\right)
\end{aligned}
\tag{4.53}
$$

The operators $A^i$ have been determined to have the following property: when acting on a highest weight state $|\psi\rangle$ of $U(4)$ with quantum numbers $(c_1, c_2, c_3, c_4)$, $A^i|\psi\rangle$

[16] For a generic primary tableaux the number of representations obtained at level $\ell$ is $\binom{4}{\ell}$ corresponding to the choice of which rows the new boxes are appended to. If the $U(4)$ highest weights of the primary are $c_1, c_2, c_3, c_4$, the representation obtained by appending new boxes to the rows $R^{i_1}$, $R^{i_\ell}$ has highest weights $c_{i_1}...c_{i_\ell}$ increased by one, while all other weights are unchanged.

is another highest weight state of $U(4)$ with quantum numbers $(c^i_1, c^i_2, c^i_3, c^i_4)$ where $c^i_j = c_j + \delta^j_i$, whenever it is well defined. The last condition (being well defined) is met if and only if the weights of $|\psi\rangle$ obey the inequality $c_i < c_{i-1}$. [17]

Let $|\psi\rangle$ denote the primary state that is a $U(4)$ highest weight. It follows that the states $A^{i_1}...A^{i_\ell}|\psi\rangle$ is the highest weight state in the representation with additional boxes in the rows $i_1...i_\ell$ described above. We will now study the norm of these states.

It is not difficult to explicitly verify that (when this state is well defined)

$$|A^i|\psi\rangle|^2 \propto (c_i - 2k - i + 1) \equiv B_i(c_i, k) \tag{4.54}$$

More generally, it is also true that

$$\left|\prod_{m=1}^{l} A^{i_m}|\psi\rangle\right|^2 \propto \prod_{m=1}^{l} B_i(c_{i_m}, k) \tag{4.55}$$

where the proportionality factor in (4.55) is a function of the the $SU(3)$ weights $c_i - c_j$ of the representation but is independent of the energy.[18] In order to see this note that different states of the form (4.55), obtained by interchanging the order of the $A^{i_m}$ operators, are each proportional to the highest weight state of a given representation. Now no $U(4)$ representation occurs more than once in the tensor product of supersymmetry generators with the primary, these representations are proportional to each other. As the commutator of $A^i$ operators is independent of energy, it follows that the proportionality factor between these states is also independent of energy.

Now the norm of the state in (4.55) clearly has a factor of $B_{i_l}(c_i)$ in it. However upon interchanging the order of the $A^i$ factors, the same result is true for $B_{i_m}$ for

[17]This is rather intuitive; when this condition is not met, the set $(c^i_1, c^i_2, c^i_3, c^i_4)$ do not constitute a valid set of labels for an irreducible representation of $U(4)$.

[18]More precisely, the proportionality factor is a function of the $c_i$ that is invariant under a uniform constant shift of each $c_i$.

each of $m = 1$ to $l$. The norm of a state at level $\ell$ is a polynomial of degree $\ell$ in the energy of the state. It follows that the full energy dependence of the norm of this state is given as in (4.55); the proportionality factor in that equation is a function only of $SU(3)$ weights and is independent of energy.

The proof presented above, strictly speaking, applies only when each of the operators $A^{i_m}$ has well defined action on $|\psi\rangle$. However, as the algebra involved in computing (4.55) is smooth (it does not care about the values of $c_i$ provided only that the state on the LHS of (4.55) is well defined), and so the result (4.55) continues to apply, whenever the state whose norm is being computed is well defined.

The unitarity restrictions and short representations of this superconformal algebra now follow almost immediately from (4.55). First consider the generic case representation where ($c_1 > c_2 > c_3 > c_4$). All states listed in (4.55) are well defined in this case and it follows $c_4 - 3 - 2k \geq 0$ is necessary and sufficient for unitarity. Representations that saturate this bound are short; the zero norm primary state is

$$|Z_4\rangle = A^4|h.w\rangle \tag{4.56}$$

consistent with the result of [14].

The state (4.55) is not well defined when $c_3 = c_4$. However even in this case the state $(A^4A^3)\,|\psi\rangle$ is well defined provided $c_2 \neq c_3$. The norm of this state is proportional to $B_4 \times B_3$. A little thought shows that the necessary and sufficient condition for unitarity is either $B_4 \geq 0$ (this is (4.56)) or that $B_3 = 0$. In the later case the representation is short, and its level one zero norm primary is $A^3|\psi\rangle$. On the other hand when $B_4 = 0$ the representation is also short. Its' zero norm primary occurs at level 2 and is $(A^4A^3)\,|\psi\rangle$.

It is clear that this pattern generalizes simply. If $c_4 = c_3 = c_2$ but $c_2 \neq c_1$ then the necessary and sufficient condition for unitarity is either $B_4 \geq 0$ or $B_3 = 0$ or $B_2 = 0$. When $B_2 = 0$ the zero norm primary occurs at level one and is given by $A^2|\psi\rangle$. When $B_3 = 0$ the zero norm primary occurs at level 2 and is given by $(A^3A^2)|\psi\rangle$. When $B_4 = 0$ the zero norm primary occurs at level 3 and is given by $(A^4A^3A^2)|\psi\rangle$.

Finally when $c_4 = c_3 = c_2 = c_1$ the necessary and sufficient condition for unitarity is either $B_4 \geq 0$ or $B_3 = 0$ or $B_2 = 0$ or $B_1 = 0$. When $B_1 = 0$ the level one primary is given by $A^1|\psi\rangle$. When $B_2 = 0$ the level two primary is given by $(A^2A^1)|\psi\rangle$. When $B_3 = 0$ the level three primary is given by $(A^3A^2A^1)|\psi\rangle$. When $B_4 = 0$ the level four primary is given by $(A^4A^3A^2A^1)|\psi\rangle$.

We may translate the analysis of zero norm states above into $SO(2) \times SO(6)$ notation by using the transformations of (4.52). This yields the result that representations are short if the energy $\epsilon_0$ and $SO(6)$ weights $h_i$ satisfy one of the following conditions (see [7, 66])

$$\begin{aligned}
\epsilon_0 =& h_1 + h_2 - h_3 + 4k + 6, \quad \text{when } \mathrm{h}_1 \geq \mathrm{h}_2 \geq |\mathrm{h}_3|. \\
\epsilon_0 =& h_1 + 4k + 4, \quad \text{when } \mathrm{h}_1 \geq \mathrm{h}_2 \text{ and } \mathrm{h}_2 = \mathrm{h}_3. \\
\epsilon_0 =& h_1 + 4k + 2, \quad \text{when } \mathrm{h}_1 = \mathrm{h}_2 = \mathrm{h}_3 \neq 0. \\
\epsilon_0 =& 4k, \quad \text{when } \mathrm{h}_1 = \mathrm{h}_2 = \mathrm{h}_3 = 0.
\end{aligned} \tag{4.57}$$

The last three conditions give isolated short representations.

### 4.3.3 Null Vectors and Character Decomposition of a Long Representation at the Unitarity Threshold

As discussed in the previous subsection, just like $d = 3$ the short representations of $d = 6$ super-conformal algebra can be broadly classified into two types, the *regular* ones and the *isolated* ones. However unlike $d = 3$ here the isolated short representations are of three kinds as we describe below. The energy of a regular short representations is given by $\epsilon_0 = h_1 + h_2 - h_3 + 4k + 6$. The null states of this representation also transform in an irreducible representation of the algebra; for $h_1 > h_2$ and $h_2 - \frac{1}{2} > |h_3 - \frac{1}{2}|$ the highest weights of the primary at the head of this (null) irreducible representation (which occurs at level 1) are given in terms of the highest weight of the representation by $\epsilon_0' = \epsilon_0 + \frac{1}{2}$, $h_1' = h_1 - \frac{1}{2}$, $h_2' = h_2 - \frac{1}{2}$, $h_3' = h_3 + \frac{1}{2}$, $k' = k + \frac{1}{2}$, $k_i' = k_i$ (where $i = 1, 2, ...., (n-1)$) and $k, k_i$ are half the weights of the R-symmetry group $Sp(2n)$ in the orthogonal basis as defined in subsection (§§4.3.1). Note that $\epsilon_0' - h_1' - h_2' + h_3' - 4k' - 6 = \epsilon_0 - h_1 - h_2 + h_3 - 4k - 6 = 0$, so that the null states also transform in a regular short representation. As union of the ordinary and null state of such short representations is identical to the state content of a long representation at the edge of the unitarity bound, we conclude that,

$$\begin{aligned}
&\lim_{\delta\to 0} \chi[h_1{+}h_2 - h_3 + 4k + 6 + \delta, h_1, h_2, h_3, k, k_i] \\
&\qquad = \chi[h_1 + h_2 - h_3 + 4k + 6, h_1, h_2, h_3, k, k_i] \\
&\qquad + \chi[h_1 + h_2 - h_3 + 4k + \frac{13}{2}, h_1 - \frac{1}{2}, h_2 - \frac{1}{2}, h_3 + \frac{1}{2}, k + \frac{1}{2}, k_i], \\
&\qquad\qquad\qquad\qquad (\text{with } \mathrm{h}_1 > \mathrm{h}_2 > |\mathrm{h}_3| \geq 0),
\end{aligned} \tag{4.58}$$

where $\chi(\epsilon_0, h_1, h_2, h_3, k, k_i)$ denotes the character of the irreducible representation of super-conformal algebra with energy $\epsilon_0$, $SO(6)$ highest weight $(h_1, h_2, h_3)$ and $Sp(2n)$ highest weight $(k, k_i)$.

On the other hand, when $h_1 > h_2 = h_3 (= h$, say) the null states of the regular short representation occur at level 2 and are labelled by a primary with highest weights $\epsilon_0' = \epsilon_0 + 1, \quad h_1' = h_1 - 1, \quad h_2' = h_2 = h, \quad h_3' = h_3 = h, \quad k' = k + 1, \quad k_i' = k_i,$ where $\epsilon_0, h_i, k, k_i$ refer to the highest weights of the original representation. Note in particular that $h_2' = h_3'$ and $\epsilon_0' - h_1' - 4k' - 4 = \epsilon_0 - h_1 - h_2 + h_3 - 4k - 6 = 0$. It follows that the null states of this representation transform in an isolated short representation and we conclude,

$$\begin{aligned}\lim_{\delta\to 0}\chi[h_1+4k+6+\delta, h_1, h, h, k, k_i] &= \chi[h_1+4k+6, h_1, h, h, k, k_i] \\ &\quad + \chi[h_1+4k+7, h_1-1, h, h, k+1, k_i] \\ &\quad (\text{with } \mathrm{h}_1 > \mathrm{h}_2 = \mathrm{h}_3 = \mathrm{h} \geq 0).\end{aligned} \tag{4.59}$$

As we have discussed earlier isolated short representations are separated from all other representations with the same $SO(6)$ and $Sp(2n)$ quantum numbers by a gap in energy. Hence it is not possible to *approach* such a representation with long representations; consequently we have no equivalent of (4.59) at energies equal to $h_1 + 4k + 7 + \delta$.

Similarly when $h_1 = h_2 = h_3 (= h \neq 0)$ the null states of the regular representation occur at level 3 and are labelled by a primary with highest weights $\epsilon_0' = \epsilon_0 + \frac{3}{2}, \quad h_1' = h - \frac{1}{2}, \quad h_2' = h - \frac{1}{2}, \quad h_3' = h - \frac{1}{2}, \quad k' = k + \frac{3}{2}$. Note in particular that $h_1' = h_2' = h_3'$ and $\epsilon_0' - h_1' - 4k' - 2 = \epsilon_0 - h_1 - 4k - 6 = 0$. Consequently the null states of this representation transforms in an isolated short representation, and we conclude,

$$\lim_{\delta\to 0}\chi[h+4k+6+\delta,h,h,h,k,k_i] =\chi[h+4k+6,h,h,h,k,k_i] + \chi[h+4k+\frac{15}{2},h-\frac{1}{2},h-\frac{1}{2},h-\frac{1}{2},k+\frac{3}{2},k_i]. \quad (\text{with } \mathrm{h_1=h_2=h_3=h>0}) \tag{4.60}$$

As explained above, we have no equivalent of (4.60) at energies equal to $h+4k+\frac{15}{2}+\delta$ which corresponds to the unitarity bound for an isolated short representation.

Finally when $h_1=h_2=h_3=0$ the null states of the regular representation occur at level 4 and are labelled by primary with highest weights $\epsilon_0'=\epsilon_0+2, \quad h_1'=h_1=0, \quad h_2'=h_2=0, \quad h_3'=h_3=0, \quad k'=k+2, \quad k_i'=k_i$. Note in particular that in this case $h_1'=h_2'=h_3'=0$ and $\epsilon_0'-4k'=\epsilon_0-4k-6=0$. Therefore the null states of this representation transform in an isolated short representation and we conclude,

$$\lim_{\delta\to 0}\chi[4k+6+\delta,0,0,0,k,k_i] = \chi[4k+6,0,0,0,k,k-i]+\chi[4k+8,0,0,0,k+2,k_i]. \quad (\text{with } \mathrm{h_1=h_2=h_3=0}) \tag{4.61}$$

There is no equivalent of (4.61) at energies equal to $4k+6+\delta$.

As in the previous section, the analysis of the character formulae above and the definition of Indices is much simplified by the introduction of some additional notation. Given a short representation we will use the notation $c(h_1,h_2,h_3,k,k_i)$ to refer to this representation where the relationship between the numbers $h_i,k,k_i$ and the highest weights of the representation in question is defined in Table 4.2.

Table 4.2: Notations for short representations

| notation for rep. | $\epsilon_0$ | $SO(6)$ highest weight | $Sp(2n)$ highest weight |
|---|---|---|---|
| **Regular Short Rep.** $c(h_1, h_2, h_3, k, k_i)$ (with $h_1 \geq h_2 \geq \lvert h_3 \rvert$ and $k \geq 0$) | $h_1 + h_2 - h_3 + 4k + 6$ | $(h_1, h_2, h_3)$ | $(k, k_i)$ |
| **Isolated Short Reps** $c(h_1, h - \frac{1}{2}, h + \frac{1}{2}, k, k_i)$ (with $h_1 \geq h + \frac{1}{2}$ $h \geq 0$ and $k \geq -\frac{1}{2}$) | $h_1 + 4k + \frac{11}{2}$ | $(h_1 - \frac{1}{2}, h, h)$ | $(k + \frac{1}{2}, k_i)$ |
| $c(h, h, h+1, k, k_i)$ (with $h \geq 0$ and $k \geq -1$) | $h + 4k + 6$ | $(h, h, h)$ | $(k + 1, k_i)$ |
| $c(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, k, k_i)$ (with $k \geq -\frac{3}{2}$ | $4k + 6$ | $(0, 0, 0)$ | $(k + \frac{3}{2}, k_i)$ |

### 4.3.4 Indices

As in the $d = 3$ case, we define an Index for $d = 6$ as any linear combination of the multiplicities of short representations that evaluates to zero on every collection of representations that appear on the RHS of (4.58), (4.59), (4.60),and (4.61) so that it is invariant under any deformation of superconformal field theory under which the spectrum evolves continuously. We now proceed to list all of these Indices,

1. The simplest Indices are given by the multiplicities of short representations in the spectrum that never appear on the RHS of (4.58), (4.59), (4.60),and (4.61) (for any values of the quantum numbers on the LHS of those equations). All such representations are easy to list; they are

- $c(h_1, h - \frac{1}{2}, h + \frac{1}{2}, k, k_i)$ for all $h_1 \geq h + \frac{1}{2}$, $h \geq 0$ and $k - k_1 = -\frac{1}{2}, 0$.
- $c(h, h, h+1, k, k_i)$ for all $h \geq 0$ and $k - k_1 = -1, -\frac{1}{2}, 0$
- $c(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, k, k_i)$ for $k - k_1 = -\frac{3}{2}, -1, -\frac{1}{2}, 0$

In all the above cases we must consider all the possible values of the set $k_i, i = 1 \ldots n-1$. This means $k_1 \geq k_2 \geq \ldots \geq k_{n-1} \geq 0$ and the $k_i$ may each be integers or half integers.

2. We can also construct Indices from linear combinations of the multiplicities of representations that do appear on the RHS of (4.58), (4.59), (4.60),and (4.61). The complete list of such linear combinations is given by,

$$I_{M_1, M_2, M_3, \{k_i\}} = \sum_{p=M_3-1}^{2(M_1-k_1)} (-1)^{p+1} n\{c(M_2 + \frac{p}{2}, \frac{p}{2}, M_3 - \frac{p}{2}, M_1 - \frac{p}{2}, k_i)\}, \quad (4.62)$$

where $n\{R\}$ denotes the number of representations of type $R$ and the Index labels $M_1$, $M_2$ and $M_3$ are respectively the values of $h_2 + k$, $h_1 - h_2$ and $M_3 = h_2 + h_3$ for the regular representations that appears in the above sum. Here $M_2$ and $M_3$ are integers greater than or equal to zero and $M_1$ is an integer or half integer with $M_1 \geq \frac{M_3}{2} + k_1$.

### 4.3.5 Minimally BPS states: distinguished supercharge and commuting superalgebra

Consider the special Q with charges ($h_1 = -\frac{1}{2}, h_2 = -\frac{1}{2}, h_3 = \frac{1}{2}, k = \frac{1}{2}, \epsilon_0 = \frac{1}{2}$). Let $S = Q^\dagger$; it is then easily verified that,

$$2\{S, Q\} \equiv \Delta = \epsilon_0 - (h_1 + h_2 - h_3 + 4k) \tag{4.63}$$

Just as in $d = 3$, we shall define a partition function over states annihilated by Q. Again all such states transform in an irreducible representation of the subalgebra of the superconformal algebra that commutes with $Q, S$ and hence $\Delta$. This subalgebra is easily determined to be the supergroup $D(3, \frac{n-2}{2})$ (see [7]).

The bosonic subgroup of this commuting superalgebra is $SU(3,1) \otimes Sp(n-2)$. The usual Cartan charges of $SU(3,1)$ and the Cartan charges of $Sp(n-2)$ are given in terms of the Cartan elements of the full superconformal algebra by,

$$E = 3\epsilon_0 + h_1 + h_2 - h_3; H_1 = h_1 - h_2; H_2 = h_2 + h_3; K_i = k_{i+1}, \tag{4.64}$$

where $E$ is the $U(1)$ Cartan, $(H_1, H_2)$ are the $SU(3)$ Cartans (in the Dynkin basis) and $K_i$ are the $Sp(n-2)$ Cartans (in the orthogonal basis). [19]

### 4.3.6 A Trace formula for the general Index and its Character Decomposition

As in the case of $d = 3$, we define the Witten Index as,

$$I^W = Tr_R[(-1)^F \exp(-\zeta\Delta + \mu G)], \tag{4.65}$$

[19] Specifically the Cartans $H_1$ and $H_2$ are the following $3 \times 3$ SU(3) matrices,

$$H_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

.

Where the trace is evaluated over any Hilbert space that hosts a representation of the $d = 6$ superconformal algebra. Here $F$ is the fermion number operator; by the spin statistics theorem, in any quantum field theory we take $F = 2h_2$. G is any element of the subalgebra that commutes with the set set $\{Q, S, \Delta\}$; by a similarity transformation , G may always be rotated in to a linear combination of the subalgebra Cartan generators.

The Witten Index (4.65) receives contributions only from the states that are annihilated by both $Q$ and $S$ (all other states yields contribution that cancel in pairs) and, hence, have $\Delta = 0$. So it is independent of $\zeta$. The usual arguments[2] also ensure that $I^W$ is also an Index and hence it should be possible to expand $I^W$ as a linear combination of the Indices defined in the previous section. In fact it is easy to check that for any representation $A$ (reducible or irreducible) of the $d = 6$ superconformal algebra,

$$\begin{aligned}
I^{wi}(A) = & \sum_{M_1,M_2,M_3,\{k_i\}} I_{M_1,M_2,M_3}\chi_{sub}(M_2, M_3, k_i, 4(M_2 - M_3) + 12M_1 + 24) \\
+ & \sum_{\{k_i\},k-k_1=-\frac{3}{2},-1,-\frac{1}{2},0} n\{c(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, k, k_i)\}\chi_{sub}(0, 0, k_i, 12k + 18) \\
+ & \sum_{\{k_i\},h\geq 0,k-k_1=-1,-\frac{1}{2},0} (-1)^{2h+1} n\{c(h, h, h + 1, k, k_i)\}\chi_{sub}(0, 2h + 1, k_i, 4h + 12k + 20) \\
+ & \sum_{\{k_i\},h_1,h(h_1\geq h\geq 0),k-k_1=-\frac{1}{2},0} \left[(-1)^{2h} n\{c(h_1, h, h + 1, k, k_i)\}\right. \\
& \left.\times\chi_{sub}(h_1 - h, 2h + 1, k_i, 4h_1 + 12k + 20)\right].
\end{aligned} \tag{4.66}$$

where $\chi_{sub}(H_1, H_2, K_i, E)$ is the supercharacter of the representation with highest weights $H_1, H_2, K_i, E$ as defined in (4.64). In the first sum in (4.66) $M_2$ and $M_3$ run over integers greater than or equal to zero and $M_1$ runs over integers or half integers with $M_1 \geq \frac{M_3}{2} + k_1$. Also the set $\{k_i\}$ runs over integer and half integer values satisfying the condition $k_1 \geq k_2 \cdots \geq k_n$. In order to obtain (4.66) we have used,

$$\begin{aligned} I^{wi}[(c(h_1, h_2, h_3, k, k_i)(\text{with } \mathrm{h}_1 \geq \mathrm{h}_2 \geq |\mathrm{h}_3| \text{ and } \mathrm{k} \geq 0)] = \\ (-1)^{2h_2+1} \chi_{sub}(h_1 - h_2, h_2 + h_3, k_i, 4(h_1 + h_2 - h_3) + 12k + 24). \end{aligned} \tag{4.67}$$

$$\begin{aligned} I^{wi}[(c(h_1, h, h+1, k, k_i)(\text{with } h_1 \geq h \geq 0 \text{ and } \mathrm{k} \geq -\frac{1}{2})] = \\ (-1)^{2h+1} \chi_{sub}(h_1 - h, 2h+1, k_i, 4h_1 + 12k + 20). \end{aligned} \tag{4.68}$$

$$\begin{aligned} I^{wi}[(c(h, h, h+1, k, k_i)(\text{with } \mathrm{h} \geq 0 \text{ and } \mathrm{k} \geq -1)] = \\ (-1)^{2h+1} \chi_{sub}(0, 2h+1, k_i, 4h + 12k + 20). \end{aligned} \tag{4.69}$$

$$I^{wi}[(c(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, k, k_i)(\text{with } \mathrm{k} \geq -\frac{3}{2})] = \chi_{\mathrm{sub}}(0, 0, \mathrm{k_i}, 12\mathrm{k} + 18). \tag{4.70}$$

Equations (4.67)-(4.70) follow from the observation that the set of $\Delta = 0$ states (the only states that contribute to the Witten Index) in any short representation of the superconformal algebra transform in a single representation of the commuting super subalgebra. The quantum numbers of these representations of the subalgebra are easily determined, given the quantum numbers of the parent short representation. In the case of regular short representations, a primary of the subalgebra representation (in which the $\Delta = 0$ states transform) is obtained by acting on the highest weight primary of the full representation (which turns out to have $\Delta = 6$) with supercharges

$Q_1, Q_2$ and $Q_3$ with quantum numbers $(h_1 = \frac{1}{2}, h_2 = \frac{1}{2}, h_3 = \frac{1}{2}, k = \frac{1}{2}, k_i = 0, \epsilon_0 = \frac{1}{2})$, $(h_1 = \frac{1}{2}, h_2 = -\frac{1}{2}, h_3 = -\frac{1}{2}, k = \frac{1}{2}, k_i = 0, \epsilon_0 = \frac{1}{2})$ and $(h_1 = -\frac{1}{2}, h_2 = \frac{1}{2}, h_3 = -\frac{1}{2}, k = \frac{1}{2}, k_i = 0, \epsilon_0 = \frac{1}{2})$ respectively, all of which have $\Delta = -2$. The Witten Index evaluated over these representations in terms of the supercharacter of the subgroup is given by (4.67).

In the case of isolated representations the highest weight primary of the full representation turns out to have $\Delta = 4, 2$ and $0$; for the $\Delta = 4$ case the primary of the subalgebra is obtained by the action of $Q_1$ and $Q_2$ on the primary of the full superconformal algebra, and for $\Delta = 2$ case it is obtained by the action of $Q_1$. The highest weight of an isolated superconformal short which itself has $\Delta = 0$ is also a primary of the commuting subalgebra. The Witten Index evaluated over these representations in terms of the supercharacter of the subgroup is given by (4.68), (4.69) and (4.70).

Note that every Index that appears in the list of subsection §§4.3.4 appears as the coefficient of a distinct subalgebra supercharacter in (4.66). As supercharacters of distinct irreducible representations are linearly independent, it follows that knowledge of $I^W$ is sufficient to reconstruct all superconformal Indices of the algebra. In this sense (4.66) is the most general Index that can be constructed from the superconformal algebra alone.

### 4.3.7 The Index over M theory multi gravitons in $AdS_7 \times S^4$

We now compute the Witten Index defined for the for the world volume theory of the $M5$ brane in the large $N$ limit. The R-symmetry for this algebra is $SO(5)$ corresponding to rotations in the 5 directions transverse to the brane. This is consistent with the formalism above because $SO(5) \sim Sp(4)$. We will use the symbols $l_1, l_2$ to represent the $SO(5)$ Cartans in the orthogonal basis. The $Sp(4)$ Cartans are given by:

$$k = \frac{l_1 + l_2}{2}, \quad k_1 = \frac{l_1 - l_2}{2} \tag{4.71}$$

Note, also that the bosonic part of the commuting subalgebra is $SU(3,1) \otimes Sp(2)$. In the calculation below, we will us the equivalence $Sp(2) \sim SU(2)$. The $SU(2)$ charge is the same as the $Sp(2)$ charge.

In the strict large $N$ limit, the spectrum of this theory is the Fock space of supergravitons of $M$ theory on $AdS_7 \times S^4$ [1, 57].[20] The set of primaries for the graviton spectrum is ($\epsilon_0 = 2p, l_1 = 2p, l_2 = 0, h_1 = 0, h_2 = 0, h_3 = 0$) [67][21], where p can be any positive integer. Now given a highest weight state, we again use the Racah Speiser algorithm to obtain the representations (of the maximal compact subgroup) occurring in the supermultiplet. The result is enumerated in table 4.3 and agrees with [67]. By the action of momentum operators on this states we can build up the entire representation of the superconformal algebra.

It is now again simple to compute the Index over single gravitons once we have the spectrum. The Witten Index for the $p^{th}$ graviton representation ($R_p$)(i.e. for a particular value of p in the primary), is obtained by

[20]The Index we will calculate is sensitive to $\frac{1}{16}$ BPS states. However, the $\frac{1}{4}$ BPS partition function has been calculated, even at finite $N$, in [58]

[21]we specify the highest weight of the maximal compact subgroup; $\epsilon_0$ being the $SO(2)$ charge, $l_1$ and $l_2$ being the $SO(5)$ charges in orthogonal basis and $h_1, h_2$ and $h_3$ being the $SO(6)$ charge also in the orthogonal basis

[22]The '+' appears because the conformal representation we subtract is, itself short. See [6] for details.

Table 4.3: d=6 graviton spectrum

| range of $p$ | $\epsilon_0[SO(2)]$ | $SO(6)$[orth.] | $SO(5)$[orth.] | $\Delta$ | contribution |
|---|---|---|---|---|---|
| $p \geq 1$ | $2p$ | $(0,0,0)$ | $(p,0)$ | 0 | $+$ |
| $p \geq 1$ | $2p+\frac{1}{2}$ | $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ | $(\frac{2p-1}{2},\frac{1}{2})$ | 0 | $+$ |
| $p \geq 1$ | $2p+1$ | $(1,1,1)$ | $(p-1,0)$ | 2 | $+$ |
| $p \geq 2$ | $2p+1$ | $(1,0,0)$ | $(p-1,1)$ | 0 | $+$ |
| $p \geq 2$ | $2p+\frac{3}{2}$ | $(\frac{3}{2},\frac{1}{2},\frac{1}{2})$ | $(\frac{(2p-3)}{2},\frac{1}{2})$ | 2 | $+$ |
| $p \geq 2$ | $2p+2$ | $(2,0,0)$ | $(p-2,0)$ | 4 | $+$ |
| $p \geq 3$ | $2p+\frac{3}{2}$ | $(\frac{1}{2},\frac{1}{2},-\frac{1}{2})$ | $(\frac{2p-3}{2},\frac{3}{2})$ | 0 | $+$ |
| $p \geq 3$ | $2p+2$ | $(1,1,0)$ | $(p-2,1)$ | 2 | $+$ |
| $p \geq 3$ | $2p+\frac{5}{2}$ | $(\frac{3}{2},\frac{1}{2},-\frac{1}{2})$ | $(\frac{(2p-5)}{2},\frac{1}{2})$ | 4 | $+$ |
| $p \geq 3$ | $2p+3$ | $(1,1,-1)$ | $(p-3,0)$ | 6 | $+$ |
| $p \geq 4$ | $2p+2$ | $(0,0,0)$ | $(p-2,2)$ | 2 | $+$ |
| $p \geq 4$ | $2p+\frac{5}{2}$ | $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ | $(\frac{2p-5}{2},\frac{3}{2})$ | 4 | $+$ |
| $p \geq 4$ | $2p+3$ | $(1,0,0)$ | $(p-3,1)$ | 6 | $+$ |
| $n \geq 4$ | $2p+\frac{7}{2}$ | $(\frac{1}{2},\frac{1}{2},-\frac{1}{2})$ | $(\frac{2p-7}{2},\frac{1}{2})$ | 8 | $+$ |
| $p \geq 4$ | $2p+4$ | $(0,0,0)$ | $(p-4,0)$ | 12 | $+$ |
| $p=1$ | $\frac{7}{2}$ | $(\frac{1}{2},\frac{1}{2},-\frac{1}{2})$ | $(\frac{1}{2},\frac{1}{2})$ | 0 | $-$ |
| $p=1$ | 4 | $(1,1,0)$ | $(0,0)$ | 2 | $-$ |
| $p=1$ | 4 | $(0,0,0)$ | $(1,0)$ | 2 | $-$ |
| $p=1$ | 5 | $(1,0,0)$ | $(0,0)$ | 4 | $+$[22] |
| $p=1$ | 6 | $(0,0,0)$ | $(0,0)$ | 6 | $-$ |
| $p=2$ | 6 | $(0,0,0)$ | $(1,1)$ | 2 | $-$ |
| $p=2$ | $\frac{13}{2}$ | $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ | $(\frac{1}{2},\frac{1}{2})$ | 4 | $-$ |
| $p=2$ | 7 | $(1,0,0)$ | $(0,0)$ | 6 | $-$ |

$$\begin{aligned} I^W_{R_p} =& Tr_{\Delta=0}\Big[(-1)^F x^E z^{K_1} y_1^{H_1} y_2^{H_2}\Big] \\ =& \sum_q \frac{(-1)^{2(h_2)_q} x^{(3\epsilon_0+h_1+h_2-h_3)_q}\chi_q^{SU(2)}(z)\chi_q^{SU(3)}(y_1,y_2)}{(1-x^4y_1)(1-\frac{x^4y_2}{y_1})(1-\frac{x^4}{y_2})}, \end{aligned} \tag{4.72}$$

where $q$ runs over all the conformal representations with $\Delta=0$ that appears in the decomposition of $R_p$ in table 4.3; $x, z, y_1$ and $y_2$ are the exponential of the chemical potentials corresponding to the subgroup charges $E, K_1, H_1$ and $H_2$ respectively as defined in (4.64); $\chi^{SU(2)}$ and $\chi^{SU(3)}$ denote the characters of the groups $SU(2)$ and

$SU(3)$ respectively, which are computed using the Weyl character formula.

The Index over the single particle states is then simply given by the following sum,

$$I^W_{sp} = \sum_{p=3}^{\infty} I^W_{R_p} + I^W_{R_2} + I^W_{R_1}, \tag{4.73}$$

Performed this sum, we find that the single particle contribution to the Index is

$$\begin{aligned} I^W_{sp} &= \frac{\text{term1} + \text{term2}}{\text{den}} \\ \text{term1} &= x^6 \left(\sqrt{z} y_1^2 \left(1 - x^8 y_2\right) x^2 + \sqrt{z} y_2 \left(1 - x^8 y_2\right) x^2\right) \\ \text{term2} &= x^6 \left(y_1 \left(-\sqrt{z} x^{10} + \sqrt{z} y_2^2 x^2 + \left(x^{12} - 1\right)(z+1) y_2\right)\right) \\ \text{den} &= \left(\sqrt{z} x^{12} - (z+1) x^6 + \sqrt{z}\right)\left(x^4 y_1 - 1\right)\left(x^4 - y_2\right)\left(x^4 y_2 - y_1\right). \end{aligned} \tag{4.74}$$

The Index over the Fock-space of gravitons can be obtained from the above single particle Index by the formula (4.18).

To get a sense for the formula, let us set $z, y_i \to 1$ in (4.74) leaving only $x \equiv e^{-\beta}$. We remind the reader that $\beta$ is the chemical potential corresponding to $E = 3\epsilon_0 + h_1 + h_2 - h_3$. This leads to

$$I^W_{sp}(x)\Big|_{z, y_i \to 1} = \frac{x^6\left(2x^4 + x^2 + 2\right)}{\left(x^8 + x^6 - x^2 - 1\right)^2}. \tag{4.75}$$

We note that in the high energy limit when $x \to 1$, $I^W_{sp}$ in (4.75) becomes $I^W_{sp} = \frac{5}{144\beta^2}$. Then by the use of (4.18) we have,

$$I^W_{fock} = \exp \frac{5\zeta(3)}{144\beta^2}. \tag{4.76}$$

Then the average value of $E = 3\epsilon_0 + h_1 + h_2 - h_3$ is given by,

$$E = -\frac{\partial \ln I^W_{fock}}{\partial \beta} = \frac{5\zeta(3)}{72\beta^3}. \tag{4.77}$$

If we define an entropy like quantity $S$ by

$$I^W_{fock} = \int dy \exp\left(-\beta y\right) \exp S^{\text{ind}}(y), \tag{4.78}$$

we find,

$$S^{\text{ind}}(E) = \frac{5\zeta(3)/48}{(5\zeta(3)/72)^{\frac{2}{3}}} E^{\frac{2}{3}}. \tag{4.79}$$

We can also do a similar analysis with the partition function instead of the Index. The single particle partition function evaluated on the $\Delta = 0$ states with all the other chemical potentials except $\beta$ set to zero is given by,

$$Z_{sp}(x) = \text{tr}_{\Delta=0} x^E = \frac{-x^6\left(-2x^8 + x^6 + x^2 - 2\right)}{\left(1-x^2\right)^5\left(x^2+1\right)\left(x^4+x^2+1\right)^2}. \tag{4.80}$$

The separate contributions of the bosonic and fermionic states to the partition function in (4.80) are as follows,

$$Z^{\text{bose}}_{\text{sp}}(x) = \text{tr}_{\Delta=0 \text{ bosons}} = \frac{x^6\left(3x^{10} - x^6 + 2\right)}{\left(1-x^4\right)^3\left(1-x^6\right)^2} \tag{4.81}$$

$$Z^{\text{fermi}}_{\text{sp}}(x) = \text{tr}_{\Delta=0 \text{ fermions}} = \frac{x^8\left(2x^{10} - x^4 + 3\right)}{\left(1-x^4\right)^3\left(1-x^6\right)^2} \tag{4.82}$$

An analysis similar to that done for the Index, yields for the above partition function

$$\ln Z_{\text{fock}} = \sum_n \frac{Z^{\text{bose}}_{\text{sp}}(x^n) + (-1)^{n+1} Z^{\text{fermi}}_{\text{sp}}}{n} = \frac{7\zeta(6)}{2048\beta^5} \tag{4.83}$$

$$S(E) = \frac{21\zeta(6)/1024}{(35\zeta(6)/2048)^{\frac{5}{6}}} E^{\frac{5}{6}}, \tag{4.84}$$

which is again similar to that of a six dimensional gas for reasons that are similar to those explained below equation (4.29). Note, that in this case, we have 2 transverse supersymmetric scalars and 3 derivatives.

### 4.3.8 The Index on the worldvolume theory of a single $M5$ brane

We will now compute our Index over the worldvolume theory of a single $M5$ brane. For this free theory, the single particle state content is just the representation corresponding to $p = 1$ in Table 4.3 of the previous subsection. This means that it corresponds to the representation of the $d = 6$ superconformal group with the primary having charges $\epsilon_0 = 2$, $SO(6)$ highest weights $[0, 0, 0]$ and R-symmetry $SO(5)$ highest weight $[1, 0]$. Physically, this multiplet corresponds to the 5 transverse scalars, real fermions transforming as chiral spinors of both $SO(6)$ and $SO(5)$ and a self-dual two form $B_{\mu\nu}$. See [68, 61, 62] and references therein for more details. Using Table 4.3, we calculate the Index over these states

$$\begin{aligned} I^{\rm sp}_{M_5}(x, z, y_1, y_2) &= \mathrm{Tr}\left[(-1)^F x^E z^{K_1} y_1^{H_1} y_2^{H_2}\right] \\ &= \frac{x^6(z^{\frac{1}{2}} + \frac{1}{z^{\frac{1}{2}}}) - x^8\left(y_2 + \frac{y_1}{y_2} + \frac{1}{y_1}\right) + x^{12}}{(1 - x^4 y_1)\left(1 - x^4\frac{y_2}{y_1}\right)\left(1 - \frac{x^4}{y_2}\right)}. \end{aligned} \tag{4.85}$$

Specializing to the chemical potentials $y_i \to 1, z \to 1$, the Index simplifies to

$$I^{\rm sp}_{M_5}(x, z = 1, y_i = 1) = \frac{2x^6 - 3x^8 + x^{12}}{(1 - x^4)^3}. \tag{4.86}$$

Multiparticling this Index, to get the Index over the Fock space on the $M_2$ brane, we find that

$$\begin{aligned} I_{M_5}(x, z = 1, y_i = 1) &= \exp\sum_n \frac{I^{\rm sp}_{M_5}(x^n, z = 1, y_i = 1)}{n} \\ &= \prod_{n_1,n_2,n_3} \frac{\left(1 - x^{8+4(n_1+n_2+n_3)}\right)^3}{\left(1 - x^{6+4(n_1+n_2+n_3)}\right)^2\left(1 - x^{12+4(n_1+n_2+n_3)}\right)}. \end{aligned} \tag{4.87}$$

At high temperatures $x \equiv e^{-\beta} \to 1$, we find

$$I_{M_5}\big|_{x\to 1, y_i = 1} = \exp\left\{\frac{\pi^2}{32\beta}\right\}. \tag{4.88}$$

The supersymmetric single particle partition function, on the other hand is given by

$$Z^{\text{sp,susy}}_{M_5}(x,z,y_1,y_2) = \text{Tr}_{\Delta=0}\left[x^E z^{K_1} y_1^{H_1} y_2^{H_2}\right] = \frac{x^6(z^{\frac{1}{2}} + \frac{1}{z^{\frac{1}{2}}}) + x^8\left(y_2 + \frac{y_1}{y_2} + \frac{1}{y_1}\right) + x^{12}}{(1-x^4 y_1)\left(1 - x^4\frac{y_2}{y_1}\right)\left(1-\frac{x^4}{y_2}\right)}. \tag{4.89}$$

In particular, setting $z, y_i = 1$, we find

$$Z^{\text{sp,susy}}_{M_5}(x, z=1, y_i=1) = \frac{2x^6 + 3x^8 + x^{12}}{(1-x^4)^3}, \tag{4.90}$$

with contributions from the bosons and fermions being

$$\begin{aligned} Z^{\text{sp,susy,bose}}_{M_5}(x) &= \text{tr}_{\Delta=0\ \text{bosons}} x^E = \frac{2x^6 + x^{12}}{(1-x^4)^3} \\ Z^{\text{sp,susy,fermi}}_{M_5}(x) &= \text{tr}_{\Delta=0\ \text{fermions}} x^E = \frac{3x^8}{(1-x^4)^3}. \end{aligned} \tag{4.91}$$

Multiparticling this result, we find

$$\begin{aligned} Z_{M_5}(x, z=1, y_i=1) &= \exp\sum_n \frac{Z^{\text{sp,susy}}_{M_5}(x^n, z=1, y_i=1)}{n} \\ &= \prod_{n_1,n_2,n_3} \frac{\left(1 + x^{8+4(n_1+n_2+n_3)}\right)^3}{\left(1 - x^{6+4(n_1+n_2+n_3)}\right)^2\left(1 - x^{12+4(n_1+n_2+n_3)}\right)}. \end{aligned} \tag{4.92}$$

At high temperatures $x \to 1$, we find that

$$Z_{M_5}(x\to 1, z=1, y_i=1) \approx \exp\left\{\frac{45\zeta(4)}{512\beta^3}\right\}. \tag{4.93}$$

## 4.4 d=5

### 4.4.1 The Superconformal Algebra and its Unitary Representations

In $d = 5$, the bosonic part of the superconformal algebra is $SO(5,2) \otimes SU(2)$. Under the $SO(5) \otimes SO(2)$ subgroup of the conformal group the supersymmetry generators $Q^i_\mu$ $i = 1, \cdots, 4$ and $\mu = \pm\frac{1}{2}$ transform as the spinors of $SO(5)$, with charge $\frac{1}{2}$ under $SO(2)$. The R-symmetry group is $SU(2)$ and $\mu$ above is an $SU(2)$ Index. We use $k$ to represent the $SU(2)$ Cartan. The $SO(5)$ Cartans in the orthogonal basis are denoted by $h_1, h_2$. We will use $\epsilon_0$ to represent the energy which is measured by the charge under $SO(2)$. To lighten the notation, we will use the same symbols to represent the eigenvalues of states under these Cartans.

With these conventions the $Qs$ have $\epsilon_0 = \frac{1}{2}, k = \pm\frac{1}{2}$ and $SO(5)$ charges:

$$\begin{aligned} Q^1 \to &(\frac{1}{2}, \frac{1}{2}), \quad Q^2 \to (\frac{1}{2}, -\frac{1}{2}) \\ Q^3 \to &(-\frac{1}{2}, \frac{1}{2}), \quad Q^4 \to (-\frac{1}{2}, -\frac{1}{2}) \end{aligned} \tag{4.94}$$

The superconformal generators $S^\mu_i$ are the conjugates of $Q^i_\mu$ and therefore their charges are the negative of the charges above.

The anticommutator between $Q$ and $S$ is given by

$$\{S^\mu_i, Q^j_\nu\} \sim \delta^\mu_\nu \left(T^j_i\right) - \delta^j_i M^\mu_\nu \tag{4.95}$$

Here $T^j_i$ and $M^\mu_\nu$ are the $SO(5,2)$ and $SU(2)$ generators respectively.

As in the previous sections, by diagonalizing this operator one can determine when a descendant of the primary will have zero norm. Performing this analysis [7], one finds that short representations can exist when the highest weights of the primary satisfy one of the following conditions

$$\begin{aligned} \epsilon_0 =& h_1 + h_2 + 3k + 4 \;\; \text{when } \mathrm{h_1} \geq \mathrm{h_2} \geq 0 \text{ and } \mathrm{k} \geq 0, \\ \epsilon_0 =& h_1 + 3k + 3, \;\; \text{when } \mathrm{h_2} = 0 \text{ and } \mathrm{k} \geq 0, \\ \epsilon_0 =& 3k, \quad \text{when } \mathrm{h_1} = \mathrm{h_2} = 0, \text{ and } \mathrm{k} \geq 0. \end{aligned} \tag{4.96}$$

The last two conditions give isolated short representations.

### 4.4.2 Null Vectors and Character Decomposition of a Long Representation at the Unitarity Threshold

As in the case of $d = 3, 6$, and as explained in the previous section the short representations of $d = 5$ are also either *regular* or *isolated*. The energy of a *regular* short representation is given by $\epsilon_0 = h_1 + h_2 + 3k + 4$. Again the null states of such a representation transform in an irreducible representation of the algebra; for $h_1 \neq 0 \neq h_2$ the highest weight of the primary at the head of this null irreducible representation is given in terms of the highest weight of the primary of the representation itself by $\epsilon_0' = \epsilon_0 + \frac{1}{2}, \quad k' = k + \frac{1}{2}, \quad h_1' = h_1 - \frac{1}{2}, \quad h_2' = h_2 - \frac{1}{2}$. We note that $\epsilon_0' - h_1' - h_2' - 3k' - 4 = \epsilon_0 - h_1 - h_2 - 3k - 4 = 0$, which shows that the null states also transform in a *regular* short representation. Thus a long representation at the edge of this unitarity bound has the same state content as the union of ordinary and null states of such a *regular* short representation. So we conclude that,

$$\begin{aligned}\lim_{\delta\to 0}\chi(h_1+h_2+3k+4+\delta,h_1,h_2,k] =&\chi(h_1+h_2+3k+4,h_1,h_2,k)\\ &+\chi(h_1+h_2+3k+\frac{9}{2},h_1-\frac{1}{2},h_2-\frac{1}{2},k+\frac{1}{2}),\\ &(\text{with } \mathrm{h}_1 \geq \mathrm{h}_2 \geq \frac{1}{2} \text{ and } \mathrm{k} \geq 0).\end{aligned} \tag{4.97}$$

where $\chi(\epsilon_0, h_1, h_2, k)$ is the character of the irreducible representation with energy $\epsilon_0$, $SO(5)$ highest weights (in the orthogonal basis) $(h_1, h_2)$ and $SU(2)$ highest weight $k$.

Now when $h_1 \geq 1, h_2 = 0$ the null states of the regular short representation occur at level two and are characterized by a primary with the highest weights $\epsilon_0' = \epsilon_0 + 1, \quad k' = k + 1, \quad h_1' = h_1 - 1, \quad h_2' = 0$. Now we note that $h_1' \neq 0, h_2' = 0$ and

$\epsilon_0' - h_1' - 3k' - 3 = \epsilon_0 - h_1 - 3k - 4 = 0$, and so we conclude that the null states of such a type of *regular* short representation transform in an *isolated* short representation. Thus for a long representation at the edge of such a unitarity bound we have,

$$\begin{aligned}\lim_{\delta\to 0}\chi(h_1+3k+3+\delta, h_1, h_2=0, k) =& \chi(h_1+3k+3, h_1, h_2=0, k)\\ &+\chi(h_1+3k+4, h_1-1, h_2=0, k+1).\\ & h_1\geq 1, k\geq 0\end{aligned} \tag{4.98}$$

Finally when $h_1 = 0 = h_2$ the null states of the *regular* short representation occur at level four and are labeled by a primary with the highest weight $\epsilon_0' = \epsilon_0 + 2, \quad k' = k+2, \quad h_1' = 0, \quad h_2' = 0$. Here we note that $h_1' = 0 = h_2'$ and $\epsilon_0' - 3k' = \epsilon_0 - 3k - 4 = 0$, which shows that the null states of this type of *regular* short representation again transforms in an *isolated* short representation but the *isolated* short representation encountered here is different from the one encountered in the previous paragraph. Thus for long representations at the edge of this unitarity bound we have,

$$\lim_{\delta\to 0}\chi(3k+\delta, h_1=0, h_2=0, k) = \chi(3k,0,0,k)+\chi(3k+2,0,0,k+2), \quad k\geq 0. \tag{4.99}$$

Thus we see that the *isolated* short representations (as defined in the previous subsection) are separated from other representations with the same $SO(5)$ and $SU(2)$ weights by a finite gap in energy so it is not possible to *approach* such representations with long representations and therefore we do not have any equivalent of (4.98) or (4.99) at energies near $h_1 = 3k+3$ (when $h_1 \geq 1, h_2 = 0$) or near $3k$ (when $h_1 = 0 = h_2$) with $k \geq 0$ in both the cases.

For use below we define the following notation. Let $c(h_1, h_2, k)$ denote a regular short representation with $SO(5)$ and $SU(2)$ highest weights $(h_1, h_2)$ and $k$ respectively, and with $\epsilon_0 = h_1 + h_2 + 3k + 4$ (when $h_1 \geq h_2 \geq 0$). We now extend this notation to include isolated short representations.

- $c(h_1, -\frac{1}{2}, k)$ with $h_1 > 0$ and $k \geq -\frac{1}{2}$ denotes the representation with SO(5) weights $(h_1 - \frac{1}{2}, 0)$ and SU(2) quantum number $k + \frac{1}{2}$ and with $\epsilon_0 = h_1 + 3k + 4$.
- $c(-\frac{1}{2}, -\frac{1}{2}, k)$ with $k \geq -\frac{3}{2}$ denotes the representation with SO(5) weights $(0, 0)$ and SU(2) quantum number $k + \frac{3}{2}$ and $\epsilon_0 = 3k + \frac{9}{2}$.

### 4.4.3 Indices

As in the previous cases of $d = 3, 6$ for $d = 5$ an Index is defined to be any linear combination of multiplicities of short representations that evaluates to zero on every collection of collection of representations that appears on the RHS of (4.97), (4.98) and (4.99).We now list these Indices.

1. The multiplicities of short representations which never appear on the R.H.S of (4.97), (4.98) and (4.99). These are $c(-\frac{1}{2}, -\frac{1}{2}, k)$ for $k = 0, -\frac{1}{2}, -1, -\frac{3}{2}$ and $c(h_1, -\frac{1}{2}, k)$ for all $h_1 > 0$ and $k = 0, -\frac{1}{2}$.

2. The complete list of Indices constructed from linear combinations of the multiplicities of representations that appear on the RHS of (4.97), (4.98) and (4.99) is given by,

$$I^{(1)}_{M_1, M_2} = \sum_{p=-1}^{2M_2} (-1)^{p+1} n\{c(M_1 + \frac{p}{2}, \frac{p}{2}, M_2 - \frac{p}{2})\}, \tag{4.100}$$

where $n\{R\}$ denotes the multiplicities of representations of type $R$, and the Index label $M_1$ and $M_2$ are the values of $h_1 - h_2 = h_1 - \frac{p}{2}$ and $h_2 + k = \frac{p}{2} + k$ for every regular representation that appears in the sum above. Here $M_1$ can be a integer greater than or equal to zero and $M_2$ is an integer or half integer greater than or equal to zero.

### 4.4.4 Minimally BPS states: distinguished supercharge and commuting superalgebra

We consider the special Q with charges $(h_1 = -\frac{1}{2}, h_2 = -\frac{1}{2}, k = \frac{1}{2}, \epsilon_0 = \frac{1}{2})$. Let $S = Q^\dagger$ then we have,

$$\Delta \equiv \{S, Q\} = \epsilon_0 - (h_1 + h_2 + 3K) \tag{4.101}$$

We are now interested in a partition function over states annihilated by this special $Q$. Such states transform in an irreducible representation of the subalgebra of the superconformal algebra that commutes with $\{Q, S, \Delta\}$. This subalgebra turns out to be $SU(2,1)$. Note that unlike $d = 3, 6$ this subalgebra is a bosonic lie algebra, and not a super lie algebra. In the subalgebra, we will label states by their weights under the Cartan elements $H_1^s, H_2^s$, which are defined in terms of the Cartans of the full algebra by:

$$H_1^s = h_1 - h_2, \quad H_2^s = \epsilon_0 + \frac{h_1 + h_2}{2}. \tag{4.102}$$

Here, $h_1, h_2$ are the Cartans of the $SO(5)$ algebra in the orthogonal basis and $\epsilon_0$ represents the charge under $SO(2)$.

### 4.4.5 A Trace formula for the general Index and its Character Decomposition

We define the Witten Index,

$$I^w = \mathrm{Tr}_R[(-1)^F \exp(-\zeta\Delta + \mu G)], \tag{4.103}$$

where the trace being evaluated over any Hilbert Space that hosts a reducible or irreducible representation of the $d = 5$ superconformal algebra. Here $G$ is any element of the subalgebra that commutes with the set $\{S, Q, \Delta\}$ and $F = 2h_1$. It is always possible to express $G$ as a linear combination of the subalgebra Cartans (as given by (4.102)) by a similarity transformation. Once again, the Witten Index is independent of $\zeta$.

It is easy to check that the Witten Index $I^W$ evaluated on any representation $A$ (reducible or irreducible) is given by,

$$\begin{aligned} I^W(A) = & \sum_{M_1,M_2} I^{(1)}_{M_1,M_2} \chi_{sub}(M_1, \frac{3}{2}M_1 + 3(M_2+2)) \\ & + \sum_{h_1(\geq\frac{1}{2});k=-\frac{1}{2},0} n\{c(h_1, -\frac{1}{2}, k)\}\chi_{sub}(h_1 + \frac{1}{2}, \frac{3}{2}h_1 + 3k + \frac{21}{4}) \\ & + \sum_{k=-\frac{3}{2},-1,-\frac{1}{2},0} n\{c(-\frac{1}{2}, -\frac{1}{2}, k)\}\chi_{sub}(0, 3k + \frac{9}{2}) \end{aligned} \tag{4.104}$$

with $\chi_{sub}(H_1^s, H_2^s)$ is the character of a representation of the subgroup, with highest weights $(H_1^s, H_2^s)$ in the conventions described above.

In order to obtain (4.104) we have used,

$$I^{wi}(c(h_1, h_2, k)) = (-1)^{2h_2+1}\chi_{sub}(h_1 - h_2, \frac{3}{2}(h_1 + h_2) + 3k + 6) \tag{4.105}$$

$$I^{wi}(c(h_1, -\frac{1}{2}, k)) = \chi_{sub}(h_1 + \frac{1}{2}, \frac{3}{2}h_1 + 3k + \frac{21}{4}) \tag{4.106}$$

$$I^{wi}(c(-\frac{1}{2},-\frac{1}{2},k)) = \chi_{sub}(0, 3k+\frac{9}{2}) \tag{4.107}$$

Note that the states with $\Delta = 0$ in any short representation (which are the states that contribute to the Witten Index), may be organized into a single irreducible representation of the subalgebra that commutes with $Q$. The quantum numbers of this subalgebra representation may be determined in terms of the quantum numbers of the parent short representation. For a *regular* short representation the primary of the full representation has $\Delta = 4$ so the highest weight state of the representation of the subalgebra is reached by acting on it with the supercharges $Q_1, Q_2, Q_3$ with the charges $(h_1 = \frac{1}{2}, h_2 = \frac{1}{2}, k = \frac{1}{2}, \epsilon_0 = \frac{1}{2})$, $(h_1 = \frac{1}{2}, h_2 = -\frac{1}{2}, k = \frac{1}{2}, \epsilon_0 = \frac{1}{2})$, $(h_1 = -\frac{1}{2}, h_2 = \frac{1}{2}, k = \frac{1}{2}, \epsilon_0 = \frac{1}{2})$. These have $\Delta = -2, -1, -1$ respectively. Similarly an *isolated* short representation of type $c(h_1, -\frac{1}{2}, k)$ with $h_1 > 0$ and $k \geq -\frac{1}{2}$ has $\Delta = 3$ and is acted upon by $Q_1$ and $Q_2$ in order to reach the highest weight state of the representation of the subalgebra. Finally the *isolated* short representations of type $c(-\frac{1}{2}, -\frac{1}{2}, k)$ with $k \geq -\frac{3}{2}$ have $\Delta = 0$ and are themselves the highest weight states of the representation of the subalgebra.

We finally note that every Index constructed in subsection §§4.4.3 appears as the coefficient of a distinct subalgebra character in (4.104). Thus $I^W$ may be used to reconstruct all superconformal Indices of the algebra which makes it the most general Index that is possible to construct from the algebra alone.

## 4.5 Discussion

In this chapter we have presented formulae for the most general superconformal Index for superconformal algebras in 3, 5 and 6 dimensions. Our work generalizes the analogous construction of an Index for four dimensional conformal field theories presented in [14].

We hope that our work will find eventual use in the study of the space of superconformal field theories in 3, 5 and 6 dimensions. It has recently become clear that the space of superconformal field theories in four dimensions is much richer than previously suspected [69]. The space of superconformal field theories in $d = 3, 5, 6$ may be equally intricate, although this question has been less studied. As our Index is constant on any connected component in the space of superconformal field theories, it may play a useful role in the study of this space.

In this chapter we have also demonstrated that the most general superconformal Index, in all the dimensions that we have studied, is captured by a simple trace formula. This observation may turn out to be useful as traces may easily be reformulated as path integrals, which in turn can sometimes be evaluated, using either perturbative techniques or localization arguments.

The two dimensional Index – the elliptic genus – has played an important role in the understanding of black hole entropy from string theory. However the four dimensional Index defined in [14] does not seem to capture the entropy of black holes in any obvious way. It would be interesting to know what the analogous situation in in 3 and 6 dimensions. It would certainly be interesting, for instance, if the Index for the theory on the world volume of the M2 of M5 brane underwent a large $N$ transition as a function of chemical potentials, to a phase whose Index entropy scales like $N^{\frac{3}{2}}$ and $N^3$ respectively. As we currently lack a computable framework for multiple M2 or M5 branes we do not know if this happens; however see [63] for recent interesting progress in this respect.

In this connection we also note that the Index for the weakly coupled Chern Simons theories studied in this chapter does undergo a large $N$ phase transition as a function of temperature. It would be interesting to have a holographic dual description of these phase transitions.

# Chapter 5

# Supersymmetric States in $\mathrm{AdS}_3/\mathrm{CFT}_2$ I : Classical Analysis

## 5.1 Introduction

In the AdS/CFT correspondence, the duality between gravity on $\mathrm{AdS}_3$ and a 2 dimensional conformal field theory has a special place. In fact, to date, almost *all* black hole entropy calculations may be formulated as calculations in $\mathrm{AdS}_3/\mathrm{CFT}_2$ (this excludes some recent numerical work [70]). Second, conformal symmetry is enhanced in 2 dimensions and this allows us to use powerful techniques from 2 dimensional conformal field theory; we will see examples in this chapter and the next.

In this chapter, we would like to repeat the studies of supersymmetric partition functions that we performed in higher dimensions for $\mathrm{AdS}_3/\mathrm{CFT}_2$. However, in contrast to the previous chapters, here we will take a mostly bulk viewpoint. Our effort will be to directly obtain the $\frac{1}{4}$ BPS partition function, at least, at low energies from the bulk; we will then compare this result with the answer from the dual CFT.

To obtain these supersymmetric partition functions from the bulk, we need to quantize string theory in $\mathrm{AdS}_3$. In general, quantizing string theory in non-trivial spacetime backgrounds remains a difficult task. However, in the past few years, some progress has been made by approaching this problem using canonical methods [71, 72, 73, 74, 49, 75, 50, 51]. The principle behind these studies is to first find all classical supersymmetric solutions of string theory in a given background. One then assumes that these solutions can be quantized independently; often this assumption can be checked against other independent calculations.

In this chapter, we perform the first part of this programme. We will parameterize all classical supersymmetric brane probes moving in several backgrounds that are related to $\mathrm{AdS}_3/\mathrm{CFT}_2$. These are (a) the extremal D1-D5 background, (b) the extremal D1-D5-P background, (c) the smooth geometries proposed in [76, 77, 78] with the same charges as the D1-D5 system and (d) global $AdS_3 \times S^3 \times T^4/K3$.

The physical significance of these backgrounds is as follows. The AdS/CFT conjecture[1, 57] relates type IIB string theory on global $AdS_3$ to the NS sector of a 1+1 dimensional CFT on its boundary. The solutions in global AdS we find below correspond to the 1/4 BPS sector of the CFT. The NS and R sectors of this CFT are related by an operation called ‘spectral flow’. Performing this operation on the supergravity solution for global $AdS$ yields the near horizon region of one of the solutions of Lunin and Mathur [77]. This corresponds to the specific Ramond ground state obtained by spectrally flowing the NS vacuum. Other Ramond vacua are described by other solutions in [77]. The zero mass BTZ black hole which is the near-horizon of the extremal D1-D5 geometry, on the other hand, has been argued to be an 'average' over all Ramond ground states.

The 'giant graviton' brane probes we find comprise D1 branes, D5 branes and bound states of D1 and D5 branes. As we make more precise in section 5.2.2 we find that these supersymmetric probes have the property that a certain Killing vector is tangent to the brane worldvolume at each point. Hence, given the shape of the brane at any one point of time, one can translate it in time along the integral curves of this Killing vector to obtain the entire brane worldvolume. The set of all solutions is parameterized by the set of all initial shapes. This simple prescription is sufficient to describe supersymmetric probes in all the backgrounds we mentioned above.

Surprisingly, we find that the symplectic structure on these classical solutions is such that we can describe all the solutions above, including supersymmetric solutions to the DBI action on the 6 dimensional D5 brane worldvolume, in a unified 1+1 dimensional framework. It is well known that the infra-red limit of the world volume theory of a bound state of D1 branes and D5 branes, in flat space, is given by a 1+1 dimensional sigma model. However, our result which we emphasize is classical, is valid in curved backgrounds and does not rely on taking the infra-red limit.

Now, the CFT and the theory of gravity both have a large set of parameters ( the exact number depends on the compact manifold). When these parameters are tuned to a particular value, we obtain the pure 'D1-D5' system; at another value of these parameters, we obtain the 'symmetric product' description of the system.

The probes that we find are supersymmetric on a codimension 4 manifold of this parameter space. In supergravity, this means that we need to set the background

NS-NS fluxes and theta angle to zero to obtain supersymmetric probes.

Now, on this submanifold, the boundary theory is known to be singular because the stack of D1 and D5 branes that make up the background can separate at no cost in energy [79]. One may wonder then, whether the probes we find are artifacts of this singularity, *i.e*, whether they merely represent breakaway D1-D5 subsystems which can escape to infinity. In global $AdS$, and in the Ramond sector solution dual to global $AdS$, this is not the case. In these geometries, for generic parameters, the 1/4 BPS giant gravitons that we describe, are 'bound' to the center of AdS and cannot escape to infinity. This indicates that they correspond to discrete states and not to states in a continuum. In the boundary theory this means that they correspond to BPS states that are *not* localized about the singularities of the Higgs branch. Averaging over the Ramond vacua to produce the zero mass BTZ black hole, however, washes out the structure of these discrete bound states and the only solutions we are left with are at the bottom of a continuum of non-supersymmetric states.

We prove that no BPS probes survive if we turn on a small NS-NS field. This is not a contradiction for it merely means that the $\frac{1}{4}$ BPS partition function jumps as we move off this submanifold of moduli space. Further investigation of this issue in the quantum theory and of protected quantities, like the elliptic genus and the spectrum of chiral-chiral primaries is left to the next chapter.

Giant gravitons in $AdS_3$ have been considered previously [80, 81, 77, 82] and it was noted that regular 1/2 BPS brane configurations exist only for specific values of the charges. These are precisely the values at which the giant gravitons we describe can escape to 'infinity' in global AdS. The moduli space of 1/4 BPS giant gravitons, however, is far richer and this is what we will concern ourselves with in this chapter.

A brief outline of this chapter is as follows. In Section 5.2, we perform a Killing spinor and kappa symmetry analysis to determine the conditions that D brane probes, in the four backgrounds above, must obey in order to be supersymmetric. Using this insight, in section 5.3 we explicitly construct supersymmetric D1 brane solutions in these backgrounds and verify that they satisfy the BPS bound. Then, in section 5.4 we show how bound states of D1 and D5 branes(represented by D5 branes with gauge fields turned on in their worldvolume) can also be described in the framework of section 5.3. In section 5.5 we discuss the effect of turning on background NS-NS fluxes. In section 5.6 we discuss the quantization of probes moving in the near horizon region of the D1-D5 background. In section 5.7, we conclude with a summary of our results and their implications.

## 5.2 Killing spinor and kappa symmetry analysis

We consider type IIB superstring theory compactified on $S^1 \times \mathcal{K}$ where $\mathcal{K}$ is $T^4$ or $K3$. We will concentrate on the case of $T^4$, unless otherwise stated. Let us parameterize $S^1$ by the coordinate $x_5$, $T^4$ by $x^6, x^7, x^8, x^9$ and the noncompact spatial directions by $x^1, x^2, x^3, x^4$. We will use coordinate indices $x^M, M = 0, 1, \ldots, 9$; $x^m, m = 1, 2, 3, 4$; $x^a$ or $x^i, a, i = 6, 7, 8, 9$. We will parameterize the 32 supersymmetries of IIB theory by two real constant chiral spinors $\epsilon_1$ and $\epsilon_2$, or equivalently by a single complex chiral spinor $\epsilon = \epsilon_1 + i\epsilon_2$.

In Section 5.2.1 we will review the preserved supersymmetries, or the Killing spinors, of the backgrounds (a) D1-D5, (b) D1-D5-P, (c) Lunin-Mathur geometries and (d) Global $\mathrm{AdS}_3 \times S^3$. In Section 5.2.2 we will describe the construction of supersymmetric probe branes, using a kappa-symmetry analysis, which preserve a certain subset of the supersymmetries of the background geometry.

### 5.2.1 Review of supersymmetry of the backgrounds

#### SUSY of D1-D5 and D1-D5-P in the Flat space approximation

We first consider the D1-D5 system, which consists of $Q_1$ D1 branes wrapped on the $S^1$ and $Q_5$ D5 branes wrapped on $S^1 \times T^4$. Let us first compute the supersymmetries of the background ignoring back-reaction. In this approximation we regard the $Q_1$ D1 branes and the $Q_5$ D5 branes as placed in flat space. The residual supersymmetries of the system can be figured out in the following way. A D1 brane wrapped on the $S^1$ preserves the supersymmetry [1]

$$\Gamma_{\hat{0}}\Gamma_{\hat{5}}\epsilon = -i\epsilon^*. \tag{5.1}$$

Similarly, a D5 brane wrapped on $S^1 \times T^4$ preserves the supersymmetry

$$\Gamma_{\hat{0}}\Gamma_{\hat{5}}\Gamma_{\hat{6}}\Gamma_{\hat{7}}\Gamma_{\hat{8}}\Gamma_{\hat{9}}\epsilon = -i\epsilon^*. \tag{5.2}$$

The above equations can be derived by considering the BPS relations arising from IIB SUSY algebra or by considering the $\kappa$-symmetry condition on the DBI description of a D1 or D5 brane. A combined system of D1 and D5 branes will therefore preserve eight supersymmetries given by $\epsilon$'s which satisfy both (5.1) and (5.2).

[1] We will denote by $\Gamma_{\hat{M}}$ the flat space Gamma-matrices satisfying $[\Gamma_{\hat{M}}, \Gamma_{\hat{N}}] = 2\eta_{\hat{M},\hat{N}}$, By contrast, Gamma matrices in a curved space, $\Gamma_M$ will defined by $\Gamma_M = \Gamma_{\hat{M}} e^{\hat{M}}_M$ where $e^{\hat{M}}$ are the vielbeins. In the flat space approximation, $\Gamma_M = \Gamma_{\hat{M}}$.

For later reference, we set up some notation. The eight residual supersymmetries of the D1-D5 system can be described as satisfying either

$$\Gamma_{\hat{6}}\Gamma_{\hat{7}}\Gamma_{\hat{8}}\Gamma_{\hat{9}}\epsilon = \epsilon, \Gamma_{\hat{0}}\Gamma_{\hat{5}}\epsilon = -\epsilon,\ \epsilon = i\epsilon^* \tag{5.3}$$

or

$$\Gamma_{\hat{6}}\Gamma_{\hat{7}}\Gamma_{\hat{8}}\Gamma_{\hat{9}}\epsilon = \epsilon, \Gamma_{\hat{0}}\Gamma_{\hat{5}}\epsilon = \epsilon,\ \epsilon = -i\epsilon^*. \tag{5.4}$$

The two conditions above are called left- and right-moving supersymmetries, respectively. Thus the D1-D5 system has (4,4) (left,right) supersymmetries.

**D1-D5-P**

If we add to the D1-D5 system $P$ units of left-moving momentum along the $S^1$, the resulting D1-D5-P system has (0,4) supersymmetry (defined by (5.4)), in the notation of the previous paragraph.[2] In the flat space limit and for non-compact $x_5$, a left-moving momentum can be seen as arising from applying an infinite boost to the D1-D5 system in the $t$-$x_5$ plane. It is easy to see that the right-moving supersymmetries are invariant under such a boost while the left-moving supersymmetries are not. Since the supersymmetry conditions are local, the argument can be extended to the case where $x_5$ is compact.

### SUSY of the full D1-D5 and D1-D5-P geometry

It has been assumed above that the $Q_1$ D1 branes and $Q_5$ D5 branes are in flat space. For $Q_1, Q_5$ large, the metric, dilaton and the RR fields get deformed. The

[2] We adopt the slightly unusual terminology that a wave rotating counterclockwise on the $S^1$ is left-moving.

modified background geometry, applying standard constructions, is given by the 'D1-D5' geometry, described in Table (5.1) (in Section 5.3.2). This geometry should be thought of as describing an 'ensemble' rather than any particular microstate of the D1-D5 system. In case of the D1-D5-P the backreacted metric is given in (5.50) (the dilaton and RR fields are given by Table (5.1)).

To analyze unbroken supersymmetries of these backgrounds and the others to follow, we need to solve the Killing spinor equations in these backgrounds. These Killing spinors were considered, in fact for a much larger class of metrics, in [83, 84]. We quote the results of this analysis here, with a very brief introduction. The details, for each case, may be found in Appendix D of [85].

In case of the D1-D5 geometry and the other geometries we consider below, the metric may always be written in terms of vielbeins, as:

$$ds^2 = -(e^{\hat{t}})^2 + (e^{\hat{5}})^2 + e^{\hat{m}}e^{\hat{m}} + e^{\hat{a}}e^{\hat{a}}. \tag{5.5}$$

The coordinate indices are as explained in the beginning of Section 5.2. The $\hat{()}$ represents a flat space index (vielbein label). Spinors are defined with respect to a specific choice of vielbeins and they transform in the spinorial representation under a $SO(1,9)$ rotation of the vielbeins. The precise form of the vielbein, in the geometries we consider, may be found in Appendix A of [85]

Finding the residual supersymmetries of a particular background amounts to solving the Killing spinor equations. The analysis in Appendix D of [85] tells us that (5.1), (5.2) continue to describe the supersymmetries of the D1-D5 geometry, while (5.4) continues to describe the supersymmetries of the D1-D5-P geometry.

**SUSY of Lunin-Mathur geometries**

It was explained in a sequence of papers [86, 87, 76, 77, 78] that the geometry of Table 5.1 should be treated as an 'average' over several allowed D1-D5 microstates. The gravity solution dual to any particular Ramond groundstate was described by Lunin and Mathur [76, 77]. The analysis of [83, 84] and Appendix D of [85] shows that even these solutions preserve the supersymmetries given by (5.1) and (5.2).

**SUSY of Global AdS$_3 \times S^3 \times T^4$**

Type IIB string theory on global AdS$_3$ is dual to the NS sector of the CFT on the boundary. If we take the geometry to be $AdS_3 \times S^3 \times T^4$, the boundary CFT has $(4, 4)$ superconformal symmetry. We will describe these supersymmetries below.

Global $AdS_3 \times S^3$ is described by the metric

$$ds^2 = -\cosh^2\rho dt^2 + \sinh^2\rho d\theta^2 + d\rho^2 + \cos^2\zeta d\phi_1^2 + \sin^2\zeta d\phi_2^2 + d\zeta^2. \tag{5.6}$$

Here, we will find the killing spinors of this background using an alternative method, due to Mikhailov [41], which is quite illuminating. The reader will find an alternate derivation in Appendix E of [85].

The metric (5.6) arises by embedding (a) $AdS_3$ in flat $R^{2,2}$ by the equations $X^{-1} = \cosh\rho\cos t,\ X^0 = \cosh\rho\sin t,\ X^1 = \sinh\rho\cos\theta,\ X^2 = \sinh\rho\sin\theta$ and (b) $S^3$ in flat $R^4$ by the equations $Y^1 = \cos\zeta\cos\phi^1$, $Y^2 = \cos\zeta\sin\phi_1$, $Y^3 = \sin\zeta\cos\phi_2$, $Y^4 = \sin\zeta\sin\phi_2$. We can therefore regard $AdS_3 \times S^3 \times T^4$ as embedded in $R^{2,10}$ as a codimension two submanifold.

Now consider $R^{2,10}$ spinors that are simultaneously real and chiral. Regard $R^{2,10}$ as a product of $R^{2,2}(\supset AdS_3)$, $R^4(\supset S^3)$, and $R^4$ (which we compactify to get the

$T^4$). The spinors now should be regarded as transforming under $SO(2,2) \times SO(4) \times SO(4)$. It is possible to consistently restrict attention to a subclass of these spinors, namely those that are chiral under the last $SO(4)$ (this is consistent because complex conjugation does not change $SO(4)$ spinor chirality). We now have a set of 16 real or 8 complex spinors. These spinors are chiral in $R^{2,6}$ as well as in $R^4$. We will denote these spinors by $\chi$.

Let us denote by $\tilde{\Gamma}_A, A = -1, 0, 1, .., 10$ the $R^{2,10}$ gamma-matrices. We define by $N_{AdS}$ the vector in $R^{2,2}$ which is normal to the AdS$_3$ submanifold and by $N_S$ the vector in $R^4$ which is the normal to $S^3$. The prescription of [41] is that the Killing spinors are given by

$$\epsilon = \left(1 + \left(\tilde{\Gamma} \cdot N_{AdS}\right)\left(\tilde{\Gamma} \cdot N_S\right)\right)\chi. \tag{5.7}$$

where $\chi$ are the $R^{2,10}$ spinors constrained as in the previous paragraph. The two normal gamma matrices are explicitly given by $\tilde{\Gamma} \cdot N_{AdS} = (X^{-1}\tilde{\Gamma}_{-1} + X^0\tilde{\Gamma}_0 + X^1\tilde{\Gamma}_1 + X^2\tilde{\Gamma}_2)$ and $\tilde{\Gamma} \cdot N_S = X^3\tilde{\Gamma}_3 + X^4\tilde{\Gamma}_4 + X^5\tilde{\Gamma}_5 + X^6\tilde{\Gamma}_6$.

### 5.2.2 Construction of supersymmetric probes

#### D1 probe in D1-D5/D1-D5-P background: flat space approximation

We first construct supersymmetric D1 brane probes in the D1-D5 background, in the approximation described in Sec 5.2.1. Consider a probe D-string executing some motion in this background.

In this subsection we demonstrate that this probe preserves all the right-moving supercharges of the background (corresponding to supersymmetry transformations (5.4)), provided its motion is such that:

**1.** The vector

$$\mathbf{n} = \frac{\partial}{\partial t} + \frac{\partial}{\partial x_5} \tag{5.8}$$

is tangent to the brane worldvolume at every point.

**2.** The brane always maintains a positive orientation with respect to the branes that make up the background.

We will first prove these statements, and then return, at the end of this subsection, to an elaboration of their meaning.

According to assumption 1 above, $\mathbf{n}$ is tangent to the worldvolume at every point. A second, linearly independent, tangent vector may be chosen at each point so that the coefficient of $\frac{\partial}{\partial t}$ is zero; making this choice this normalized vector may be written as $\mathbf{v_2} = \sin\alpha \frac{\partial}{\partial x_5} + \cos\alpha\ \mathbf{u}$ where $\mathbf{u}$ represents a spacelike unit vector orthogonal to $x_5$. By assumption 2, we have $\sin\alpha > 0$[3]. In general the direction of $\mathbf{u}$ and the value of $\alpha$ will vary as a function of world volume coordinates. Although $\mathbf{n}, \mathbf{v_2}$ are linearly independent, they are not an orthonormal set since $\mathbf{n}$ is a null vector. We can construct an orthonormal basis of vectors $\mathbf{v_1}, \mathbf{v_2}$ at each point of the world volume by the Gram-Schmidt method, yielding

$$\mathbf{v_1} = \mathbf{n}/\sin\alpha - \mathbf{v_2} = 1/\sin\alpha \left( \frac{\partial}{\partial t} + \cos^2\alpha \frac{\partial}{\partial x_5} - \cos\alpha \sin\alpha\ \mathbf{u} \right). \tag{5.9}$$

For the probe to preserve some supersymmetry $\epsilon$ we must have, at each point of the world-volume,

$$\Gamma_{\mathbf{v_1}} \Gamma_{\mathbf{v_2}} \epsilon = -i\epsilon^*. \tag{5.10}$$

[3]When $\sin\alpha$ is less than zero the $\mathbf{v_1}$ and $\mathbf{v_2}$ are not appropriately oriented. Also $\alpha \neq 0$, because in that case, the determinant of the induced worldsheet metric would vanish.

The above equation is equivalent to

$$\left[\Gamma_{\hat{0}}\Gamma_{\hat{5}} - \frac{\Gamma_{\mathbf{u}}}{\sin\alpha}\left(\cos\alpha\Gamma_{\hat{0}} + (\sin^2\alpha\cos\alpha + \cos^3\alpha)\Gamma_{\hat{5}}\right)\right]\epsilon = -i\epsilon^*. \tag{5.11}$$

This is clearly satisfied by spinors that satisfy (5.4) since (5.4) implies that $\Gamma_0\Gamma_5\epsilon = \epsilon$ which ensures $\Gamma_0\epsilon = -\Gamma_5\epsilon$ and a consequent vanishing of the coefficient of $\Gamma_{\mathbf{u}}$ above. Note that in flat space the $\Gamma_{\hat{M}} = \Gamma_M$.[4]

The conditions 1 and 2, listed at the beginning of this subsection are easily solved by choosing a world-sheet parameterization in terms of coordinates $\sigma, \tau$, such that

$$x^M = \mathbf{n}^M\tau + x^M(\sigma),$$
$$x^0 = \tau, x_5 = x_5(\sigma) + \tau, x^q = x^q(\sigma), q = 1,2,3,4,6,7,8,9 \tag{5.12}$$

where $x_5(\sigma), x^q(\sigma)$ are arbitrary functions, except that $\partial_\sigma x_5 > 0$. To connect with the earlier discussion, we identify $\mathbf{v_2}$ as the unit vector along $\mathbf{s}^M \equiv \partial_\sigma x^M$. Note that by condition (2) above we need $\partial_\sigma x_5 = (\mathbf{n}, \mathbf{s}) > 0$ which is equivalent to our earlier condition $\sin\alpha > 0$. This constraint together with the periodicity of configurations in $\sigma$, implies that $\int d\sigma x_5(\sigma) = 2\pi R w$, where $R$ is the radius of the $x_5$ circle, and $w$ is a positive integer that we will refer to as the winding number. The configurations described in this paragraph are easy to visualize. They consist of D-strings with arbitrary transverse profiles, winding the $x_5$ direction $w$ times, and moving bodily at the speed of light in the positive $x_5$ direction.

Eqn. (5.10) is equivalent to the $\kappa$-symmetry projection, which can alternatively

[4] This derivation does not work for left-moving supercharges where (5.3) implies $\Gamma_0\epsilon = +\Gamma_5\epsilon$. Left moving supercharges are symmetries for D1-branes that move at the speed of light to the left (branes whose tangent space includes $(1, -1, 0, \ldots 0)$).

be written as

$$\Gamma\epsilon = i\epsilon^*, \quad \Gamma := \tfrac{1}{2}\Gamma_{MN}\partial_\alpha x^M \partial_\beta x^N \epsilon^{\alpha\beta}/\sqrt{-h}$$
$$= \tfrac{1}{2}[\Gamma_{\mathbf{n}}, \Gamma_{\mathbf{s}}]/\sqrt{-h} = \Gamma_{\mathbf{v_1}}\Gamma_{\mathbf{v_2}}, \tag{5.13}$$

where $h$ is the determinant of the induced metric on the world volume in the $\sigma, \tau$ coordinates above. In the second line we have used the parameterization (5.12). This is equivalent to (5.10) by using $\sqrt{-h} = \sin\alpha|\mathbf{s}|$.

Since all we needed in the above discussion is the (0,4) supersymmetry (5.4) of the background, the above discussion goes through unchanged for D1 probes in the D1-D5-P background in the flat space approximation.

### D1 probe in D1-D5/D1-D5-P background

We now consider the curved D1-D5-P background, described in (5.50). The specialization to the D1-D5 background is straightforward (we just need to put $r_p = 0$). We will show that (5.12), or equivalently, the condition that $\mathbf{n} = \partial_t + \partial_5$ is tangent to the world volume, again ensures the appropriate supersymmetry of the probe. For this, we need to show that (5.13) is valid in this background. We find that (see, (5.45))

$$\sqrt{-h} = \dot{X}\cdot X' \equiv \mathbf{n}\cdot\mathbf{s} = x_5'(g_{05} + g_{55}),$$
$$\Gamma\epsilon = 1/(2\sqrt{-h})[\Gamma_{\mathbf{n}}, \Gamma_{\mathbf{s}}]\epsilon = \tfrac{1}{(g_{05}+g_{55})x_5'}\left(\Gamma_{05}x_5' + (\Gamma_0 + \Gamma_5)\Gamma_q x_q'\right)\epsilon. \tag{5.14}$$

To show that $\Gamma\epsilon = \epsilon$ we need

$$\Gamma_0\epsilon = -\Gamma_5\epsilon,$$
$$(g_{05} + g_{55})^{-1}\left(\Gamma_0\Gamma_5\right)\epsilon = \epsilon. \tag{5.15}$$

The first line is equivalent to

$$e_0^{\hat{0}}\Gamma_{\hat{0}}\epsilon = -\left(e_5^{\hat{0}}\Gamma_{\hat{0}} + e_5^{\hat{5}}\Gamma_{\hat{5}}\right)\epsilon. \tag{5.16}$$

After explicitly inserting the vielbeins using equations (5.134) and (5.135) we are left with

$$\Gamma_{\hat{0}}\epsilon = -\Gamma_{\hat{5}}\epsilon, \tag{5.17}$$

which is equivalent to $\Gamma_{\hat{0}}\Gamma_{\hat{5}}\epsilon = \epsilon$. The second line of (5.15) gives rise to the same condition

$$\Gamma_{\hat{0}}\Gamma_{\hat{5}}\epsilon = \epsilon, \tag{5.18}$$

by using $e_0^{\hat{0}}e_5^{\hat{5}} = g_{05} + g_{55}$.

Thus, we have shown that a D1 brane probe moving such that $\mathbf{n} = \partial_t + \partial_5$ is always tangent to the world-volume, equivalently satisfying Eqn. (5.12), preserves the supersymmetry (5.4).

**D1 probe in Lunin-Mathur background**

We now show that the same condition as in the previous subsection, namely that $\mathbf{n}$ should be everywhere tangent to the world-volume of the D1 brane (alternatively, that the D1 brane embedding can be expressed as in (5.12)) is valid for supersymmetry of D1 probes in the background (5.52), discussed in Section 5.2.1 above. This analysis is fairly similar to the one above. In this case, Eqn. (5.14) changes to

$$\sqrt{-h} = \dot{X}\cdot X' \equiv \mathbf{n}\cdot\mathbf{s} = x_5' g_{55} + x_m'(g_{0m} + g_{5m}). \tag{5.19}$$

Hence

$$\begin{aligned}
\Gamma\epsilon &= 1/(2\sqrt{-h})[\Gamma_{\mathbf{n}}, \Gamma_{\mathbf{s}}]\epsilon \\
&= (x_5' g_{55} + x_m'(g_{0m} + g_{5m}))^{-1} \left(\Gamma_{05} x_5' + \frac{1}{2} x_q' [(\Gamma_0 + \Gamma_5), \Gamma_q]\right) \epsilon \\
&= (x_5' g_{55} + x_m'((g_{0m} + g_{5m}))^{-1} \\
&\times \ \left(\Gamma_{\hat{0}\hat{5}} x_5' g_{55} + x_q'((g_{0q} + g_{5q}) - \Gamma_q(\Gamma_0 + \Gamma_5)\right) \epsilon) \qquad (5.20)
\end{aligned}$$

Thus, if $\Gamma_{\hat{0}\hat{5}}\epsilon = \epsilon$, as in (5.4), (which also implies $(\Gamma_0 + \Gamma_5)\epsilon = 0$, using $e_0^{\hat{0}} = e_5^{\hat{5}}$), the expression (5.20), evaluates to $\Gamma\epsilon = \epsilon$. For spinors satisfying (5.4) this also implies $\Gamma\epsilon = i\epsilon^*$ which is the kappa-symmetry projection condition. In the last step of (5.20) we have used

$$\Gamma_{05} = g_{55}\Gamma_{\hat{0}\hat{5}},\ \frac{1}{2}[\Gamma_0+\Gamma_5, \Gamma_m] = \frac{1}{2}\{\Gamma_0+\Gamma_5, \Gamma_m\} - \Gamma_m(\Gamma_0+\Gamma_5) = (g_{0m}+g_{5m}) - \Gamma_m(\Gamma_0+\Gamma_5)$$

**D1 probe in Global $AdS_3 \times S^3$**

We will use the description of supersymmetries of the background as in Section 5.2.1. We will show in this section that D1 strings with world volumes, to which

$$\mathbf{n} = \partial_t + \partial_\theta + \partial_{\phi_1} + \partial_{\phi_2} \qquad (5.21)$$

is everywhere tangent, preserve 4 supercharges.

We will first mention the geometric significance of $\mathbf{n}$. Let us group the $R^{2,6}$ (see Section 5.2.1) coordinates into complex numbers as $X^{-1} + iX^0$, $X^1 + iX^2$, $Y^1 + iY^2$, $Y^3 + iY^4$. This defines a complex structure $I$ on $R^{2,6}$. In Section 5.2.1, we have defined $N_{AdS}$ as the normal to $AdS_3$ in $R^{2,2}$ and $N_S$ as the normal to $S^3$ in $R^4$. It is easy to check that the complex partner of $N_{AdS}$ is $I(N_{AdS}) = -\partial_t - \partial_\theta$, which generates (twice) the right-moving conformal spin $2h_r$. Similarly, the complex partner of $N_S$ is $I(N_S) = \partial_{\phi^1} + \partial_{\phi_2}$, which generates (twice) the $z$ component of angular momentum in the right moving $SU(2)$ (out of $SO(4) = SU(2) \times SU(2)$). The vector $\mathbf{n}$ therefore generates, $-2(h_r - J_r)$. [5].

Note, first, that $\mathbf{n}$ is a null vector (its two components are, respectively, unit timelike and unit spacelike vectors). Let $\mathbf{n}_s = K(\partial_\theta + \partial_{\phi_1} + \partial_{\phi_2})$ (the purely spatial component of $\mathbf{n}$) with the normalization $K$ chosen to give $\mathbf{n}_s$ unit norm. Consider a positively oriented purely spatial vector $\mathbf{v}_2$ at a particular point $p$ on the string at constant time. We may decompose $\mathbf{v}_2$ as

$$\mathbf{v}_2 = \sin\alpha \mathbf{n}_s + \cos\alpha \mathbf{u}, \tag{5.22}$$

where $\mathbf{u}$ is some purely spatial unit vector orthogonal to $\mathbf{n}_s$. Let us assume that the string evolves in time so that the vector $\mathbf{n}$ is always tangent to its world volume. It follows that, at the point $P$, the world volume of the string is spanned by $\mathbf{n}$ and $\mathbf{v_2}$. These two vectors are not orthogonal, but it is easy to check that with

$$\mathbf{v_1} = \frac{\mathbf{n}}{\sin\alpha} - \mathbf{v_2}, \tag{5.23}$$

$\{\mathbf{v_1}, \mathbf{v_2}\}$ form an orthonormal set, with the first vector timelike. The D-string preserves those supersymmetries of (5.7), that satisfy:

$$\tilde{\Gamma}_{\mathbf{v_1}}\tilde{\Gamma}_{\mathbf{v_2}}\epsilon = \epsilon. \tag{5.24}$$

Before proceeding further, let us introduce some terminology. Consider a complex vector $u$, say $X_1 + iX_2$. A spinor that is annihilated by $\tilde{\Gamma}_u$ is said to have spin $-$

[5]It is not difficult to check that $2h_L - 2j_L$ is generated by the vector field $\mathbf{n}' = -\partial_t + \partial_\theta - \partial_{\phi_1} + \partial_{\phi_2}$.

under rotation in the $X_1$-$X_2$ plane, while a spinor annihilated by $\tilde{\Gamma}_{\bar{u}}$ has positive spin (consequently, the spin operator is $i\tilde{\Gamma}_1\tilde{\Gamma}_2$), with similar definitions for the other directions. Let us now consider constant spinors $\chi$ whose spins(eigenvalues under this 'spin' operator) in $R^{2,2}$ and $R^4$, respectively, are $(++)(--)$ or $(--)(++)$. The spins in $T^4$ could be either $(++)$ or $(--)$ – this gives a total of 4 spinors – or two sets of complex conjugate pairs of spinors. We will now demonstrate that any giant graviton whose world volume tangent space contains the vector (5.21) preserves all 4 of these supersymmetries.

To avoid cluttering the notation below, we define:

$$\tilde{\Gamma}_{AdS} = \tilde{\Gamma}\cdot N_{AdS}, \quad \tilde{\Gamma}_S = \tilde{\Gamma}\cdot N_S, \quad \tilde{\Gamma}_{I(N_{AdS})} = \tilde{\Gamma}\cdot I(N_{AdS}), \quad \tilde{\Gamma}_{I(N_S)} = \tilde{\Gamma}\cdot I(N_S). \tag{5.25}$$

Now consider

$$\begin{aligned} A &= (\tilde{\Gamma}_{\mathbf{v_1}}\tilde{\Gamma}_{\mathbf{v_2}} - 1)(1+\tilde{\Gamma}_{AdS}\tilde{\Gamma}_S)\chi \\ &= \left(\frac{1}{\sin\alpha}\tilde{\Gamma}_{\mathbf{n}} - \tilde{\Gamma}_{\mathbf{v_2}}\right)\tilde{\Gamma}_{\mathbf{v_2}}\left(1+\tilde{\Gamma}_{AdS}\tilde{\Gamma}_S\right)\chi - \left(1+\tilde{\Gamma}_{AdS}\tilde{\Gamma}_S\right)\chi \\ &= -\frac{1}{\sin\alpha}\tilde{\Gamma}_{\mathbf{v_2}}\tilde{\Gamma}_{\mathbf{n}}\left(1+\tilde{\Gamma}_{AdS}\tilde{\Gamma}_S\right)\chi \\ &= -\frac{1}{\sin\alpha}\tilde{\Gamma}_{\mathbf{v_2}}\tilde{\Gamma}_{I(N_S)}\left[\left(1+\tilde{\Gamma}_{I(N_S)}\tilde{\Gamma}_{I(N_{AdS})}\right)\left(1+\tilde{\Gamma}_{AdS}\tilde{\Gamma}_S\right)\right]\chi \\ &= -\frac{1}{\sin\alpha}\tilde{\Gamma}_{\mathbf{v_2}}\tilde{\Gamma}_{I(N_S)}\left(1+\tilde{\Gamma}_{I(N_S)}\tilde{\Gamma}_{I(N_{AdS})}\right)\left[1+\tilde{\Gamma}_{I(N_S)}\tilde{\Gamma}_{I(N_{AdS})}\tilde{\Gamma}_{AdS}\tilde{\Gamma}_S\right], \end{aligned} \tag{5.26}$$

where we have used $\tilde{\Gamma}^2_{I(N_S)} = 1 = -\tilde{\Gamma}^2_{I(N_{AdS})}$.

It is now relatively simple to check that (5.26) vanishes when $\chi$ is any of the four spinors $(++)(--)(++)$, $(++)(--)(--)$, $(--)(++)(++)$, $(--)(++)(--)$. [6] Recall that a positive spin is annihilated by $\tilde{\Gamma}_S - i\tilde{\Gamma}_{I(N_S)}$ and by the equivalent $AdS$

[6]The first and second of these spinors are $Qs$ while the third and fourth of these are complex conjugate $Ss$.

expression. Using $\tilde{\Gamma}_S^2 = -\tilde{\Gamma}_{AdS}^2 = 1$ we find

$$\begin{aligned}
\tilde{\Gamma}_{AdS}\tilde{\Gamma}_{I(N_{AdS})}\chi_{(++)(..)} &= +i\chi_{(++)(..)}, \\
\tilde{\Gamma}_{S}\tilde{\Gamma}_{I(N_{S})}\chi_{(..)(++)} &= -i\chi_{(..)(++)}, \\
\tilde{\Gamma}_{AdS}\tilde{\Gamma}_{I(N_{AdS})}\chi_{(--)(..)} &= -i\chi_{(--)(..)}, \\
\tilde{\Gamma}_{S}\tilde{\Gamma}_{I(N_{S})}\chi_{(..)(--)} &= +i\chi_{(..)(--)},
\end{aligned} \tag{5.27}$$

from which (5.24) follows for all the spinors listed above.

We conclude that any D1 brane world volume, to which the vector **n** is always tangent, preserves the 4 supersymmetries listed above. The same is true of a D5-brane world volume that wraps the 4-torus.

### D1-D5 bound state probe

Now, we consider D5 branes that wrap the 4-torus, and move so as to keep the vector **n** tangent to their worldvolume at all points, but also have gauge fields on their worldvolume. These gauge fields, in a configuration with non-zero instanton number, can represent bound states of D1 and D5 branes. Our analysis here is valid for all four backgrounds considered above.

Consider a D5 brane with a non-zero 2-form BI field strength $F$, that wraps the $S^1 \times T^4$. We denote the world-volume coordinates by

$$\sigma^\alpha = \sigma^{1,2,6,7,8,9} \equiv \{\tau, \sigma, z^1, z^2, z^3, z^4\}.$$

The embedding of the world volume, as before, will be denoted by $x^M(\sigma^\alpha)$ and the induced metric, by $h_{\alpha\beta} = G_{MN}\partial_\alpha X^M \partial_\beta X^N$. For a non-degenerate world-volume (det $h \neq 0$) the tangent vectors $\partial_\alpha x^M$ are linearly independent and provide a basis for the tangent space at each point of the world-volume. It is clearly possible to introduce an orthonormal (in the spacetime metric $G_{MN}$) basis of six vectors $\mathbf{v}_{\hat{\alpha}}$, related to the $\partial_\alpha x^M$ by $\partial_\alpha x^M = e_\alpha^{\hat{\alpha}} \mathbf{v}_{\hat{\alpha}}$ such that

$$G_{MN} \mathbf{v}_{\hat{\alpha}}^M \mathbf{v}_{\hat{\beta}}^N = \tilde{\eta}_{\hat{\alpha}\hat{\beta}}.$$

The invertible matrix $e_\alpha^{\hat{\alpha}}$ defines 6-beins of the induced metric:

$$\begin{aligned} h_{\alpha\beta} \quad &= G_{MN} \partial_\alpha X^M \partial_\beta X^N \\ &= G_{MN} e_\alpha^{\hat{\alpha}} e_\beta^{\hat{\beta}} \mathbf{v}_{\hat{\alpha}}^M \mathbf{v}_{\hat{\beta}}^N = \tilde{\eta}_{\hat{\alpha}\hat{\beta}} e_\alpha^{\hat{\alpha}} e_\beta^{\hat{\beta}}. \end{aligned} \tag{5.28}$$

Here $\tilde{\eta}$ is 6 dimensional and $\alpha, \beta$ run over the worldvolume coordinates. We will define below

$$\gamma_{\hat{\alpha}} = \mathbf{v}_{\hat{\alpha}}^M \Gamma_M.$$

We take $\mathbf{v}_1, \mathbf{v}_2$ to be the same as in the previous subsections. The other four vectors point along the internal manifold, $\mathbf{v}_i \propto \frac{\partial}{\partial x^i}, i = 6, 7, 8, 9$.

The condition for branes with worldvolume gauge fields to be supersymmetric was considered in [88, 89]. Using the two component notation for spinors

$$\epsilon = \left( \begin{array}{c} \epsilon_1 \\ \\ \epsilon_2 \end{array} \right), \tag{5.29}$$

the BPS condition is (see Eqn. (13) of [89])

$$\begin{aligned} &\mathbf{R}\gamma_{\hat{1}\hat{2}\hat{6}\hat{7}\hat{8}\hat{9}}\epsilon = \epsilon, \\ &\mathbf{R} = \frac{1}{\sqrt{-\det\{\tilde{\eta}_{\hat{\alpha}\hat{\beta}} + F_{\hat{\alpha}\hat{\beta}}\}}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} \gamma^{\hat{\alpha}_1\hat{\beta}_1 \ldots \hat{\alpha}_n\hat{\beta}_n} F_{\hat{\alpha}_1\hat{\beta}_1} \ldots F_{\hat{\alpha}_n\hat{\beta}_n} \sigma_3^{n+1} i\sigma_2, \end{aligned} \tag{5.30}$$

where we have expressed the world-volume gauge fields in the local orthonormal frame: $F_{\alpha\beta} = F_{\hat{\alpha}\hat{\beta}} e_\alpha^{\hat{\alpha}} e_\beta^{\hat{\beta}}$. Note, that the product in (5.30), terminates at $n = 3$ because the indices are anti-symmetrized. From the $n = 0$ term we find, using the analysis of the previous subsections that the condition (5.30) can be met only for spinors that obey (5.4). The spinors (5.4) are eigenspinors of $\sigma_1$. Since $i\sigma_2$ appears in the $n = 1$ term, this term must vanish. Hence, the gauge fields must be of the form

$$F_{\hat{1}\hat{2}} = 0, \quad F_{\hat{1}\hat{i}} = -F_{\hat{2}\hat{i}}, \quad F_{\hat{i}\hat{j}} = \epsilon_{\hat{i}\hat{j}}^{\hat{k}\hat{l}} F_{\hat{k}\hat{l}}. \tag{5.31}$$

For a gauge field of this kind, the determinant above is calculated in (5.75) and

$$\sqrt{-\det\{\tilde{\eta}_{\hat{\alpha}\hat{\beta}} + F_{\hat{\alpha}\hat{\beta}}\}} = 1 + \frac{F_{\hat{i}\hat{j}}F^{\hat{i}\hat{j}}}{4}.$$

The $n = 2$ term gives us the right factor in the numerator to cancel this and the $n = 3$ term vanishes as a virtue of (5.31).

In the world-volume curved basis, our result implies (see (5.9), (5.23)) that

$$F = F_{\sigma i} d\sigma \wedge dx^i + \frac{1}{2} F_{ij} dx^i \wedge dx^j, \tag{5.32}$$

and is self-dual on the torus, i.e

$$F_{ij}\epsilon^{ij}_{\ \ kl} = F_{kl}. \tag{5.33}$$

For 'wavy instantons' where the gauge fields depend on $\sigma$ and the field strength is of the form (5.32), the Gauss law and equation (5.33) are enough to gaurantee that $F$ solves the equations of motion [90].

The form of $F$ in (5.32) is adequate to guarantee supersymmetry in all the four backgrounds considered previously. For the sake of completeness, we mention that the explicit embedding of the D5 brane in spacetime is described by the functions $X^M(\tau, \sigma, z^{1\ldots4})$ satisfying

$$\frac{\partial X^M(\tau, \sigma, z^{1\ldots4})}{\partial \tau} = \mathbf{n}^M. \tag{5.34}$$

In the coordinate systems that we will discuss, $\mathbf{n}^M$ is a constant and in such a coordinate system we again have:

$$X^M(\tau, \sigma, z^{1...4}) = X^M(\sigma) + \mathbf{n}^M \tau. \tag{5.35}$$

Using the value of $\mathbf{n}$ (5.8) in the D1-D5, D1-D5-P and Lunin-Mathur geometries, the above equation translates to:

$$t = \tau, \quad x_5 = x_5(\sigma) + \tau, \quad x^m = x^m(\sigma), \quad x^6 = z^1, \ \ldots\ , \quad x^9 = z^4, \tag{5.36}$$

while, in global $AdS_3 \times S^3 \times T^4$, using (5.21), the brane motion is:

$$t = \tau, \quad \theta = \theta(\sigma) + \tau, \quad \rho = \rho(\sigma), \quad \zeta = \zeta(\sigma), \quad \phi_1 = \phi_1(\sigma) + \tau, \quad \phi_2 = \phi_2(\sigma) + \tau,$$
$$x^6 = z^1, \ \ldots\ , \quad x^9 = z^4. \tag{5.37}$$

We are assuming, in the embedding above, that the brane wraps the internal manifold only once. The case of multiple wrapping is identical to the case of multiple brane probes, each wrapping the internal manifold once and is discussed in more detail in Section 5.4.3.

The field strength above gives rise to an induced D1 charge, $p$, on the D5 brane worldvolume, which is proportional to the second Chern class and is given by

$$p = \frac{1}{(2\pi\sqrt{\alpha'})^4} \int_{T^4} \frac{\mathrm{Tr}\,(F \wedge F)}{2}, \tag{5.38}$$

and also to an induced D3 brane charge on the 2 cycles of the $T^4$ (which we denote by $C_2$ below), proportional to the first Chern class, given by

$$p^3_{C_2} = \frac{1}{(2\pi\sqrt{\alpha'})^2} \int_{C_2} \mathrm{Tr}(F). \tag{5.39}$$

This D5 brane configuration with worldvolume gauge fields then represents a D1-D3-D5 bound state. This bound state has the property that whenever we wrap a D3 brane on a two-cycle, we need to put an equal amount of D3 brane charge on the dual two-cycle. It may be surprising that a probe of this kind, with induced D3 brane charge, is mutually supersymmetric with the D1-D5 background.

However, this fact may be familiar to the reader from another perspective. Consider a configuration of $Q_1$ D1 branes, $Q_5$ D5 branes, $Q_3$ D3 branes and $Q_3'$ D3' branes, wrapping the $5, 56789, 567, 589$ directions respectively. Following the standard BPS analysis, of say Chapter 13 in [91], the BPS bound for this configuration is:

$$M \geq \sqrt{(Q_1 + Q_5)^2 + (Q_3 - Q_3')^2}. \tag{5.40}$$

When $Q_3 = Q_3'$, this bound becomes $M \geq Q_1 + Q_5$ and it may further be shown that this configuration preserves the same supersymmetries as the D1-D5 system.

Nevertheless, we will not be interested in probes with non-vanishing first Chern class in this chapter. The AdS/CFT conjecture requires us to sum over all geometries with fixed boundary conditions for the fields at $\infty$. When we consider a D1 or D5 probe, we can reduce the D1 or D5 charge in the background so that the total D1 and D5 charge remains constant at $\infty$. A probe with non-vanishing $p^3_{C_2}$ will lead to some finite D3 charge at $\infty$ and turning on an anti-D3 charge in the background will render the probe non-supersymmetric. So, such probes must be excluded from a consideration of the supersymmetric excitations of the pure D1-D5 system. Henceforth, we will set $p^3_{C_2}$ to zero on all 2-cycles $C_2$ of the $T^4$.

## 5.3 Charge Analysis: D strings

From the Killing spinor analysis above, we conclude that in all the four different backgrounds we will consider, D-strings that move so as to keep a particular null Killing vector field tangent to their worldvolume at each point preserve 4 supersymmetries. This means, as we mentioned, that given the initial shape of the D-string we can translate it along the integral curves of this vector field to generate the entire worldvolume. In this section, we will use this fact to explicitly parameterize all supersymmetric D-string probes in terms of their initial profile functions. We will then use the DBI action to calculate the spacetime momenta of these configurations and verify the saturation of the BPS bound.

In the first subsection below, we present a general formalism that is applicable to all the examples we consider. We then proceed to apply this formalism to the extremal D1-D5 background, the D1-D5-P background, the smooth geometries of [76] and finally global AdS.

### 5.3.1 Supersymmetric D1 Probe Solutions

We introduce coordinates, $\tau$ and $\sigma$, on the D1 brane worldvolume. We use $X^M(\sigma, \tau)$ to describe the embedding of the worldsheet in spacetime, with $t \equiv X^0$ denoting time. We will use $\dot{X}^M \equiv \frac{\partial X^M}{\partial \tau}$ and $(X^M)' \equiv \frac{\partial X^M}{\partial \sigma}$. The special null vector, discussed above, is denoted by $\mathbf{n}^M$ (see also Section 5.2.2). We will always work with the string frame metric $G_{MN}$. This is the metric we use while calculating dot products. For example, $X' \cdot X' = G_{MN} X'^M X'^N$. The Ramond-Ramond 3 form field strength is denoted by $G^{(3)}_{MNP}$ and the 2 form potential is denoted by $C^{(2)}_{MN}$. The dilaton is $\phi$. The induced worldsheet metric is $h_{\alpha\beta} = G_{MN}\partial_\alpha X^M \partial_\beta X^N$. In all the cases that we consider in this section, the $NS$-$NS$ two form is set to zero.

With this notation, the bosonic part of the D1 brane action is:

$$S = \int \mathcal{L}_{\text{brane}} d\sigma d\tau = -\frac{1}{2\pi\alpha'}\int e^{-\phi}\sqrt{-h}\, d\sigma d\tau + \frac{1}{2\pi\alpha'}\int C^{(2)}_{MN}\partial_\alpha X^M \partial_\beta X^N \frac{\epsilon^{\alpha\beta}}{2} d\sigma d\tau, \tag{5.41}$$

where

$$h = \text{Det}[h_{\alpha\beta}] = (X'\cdot X')(\dot{X}\cdot\dot{X}) - (X'\cdot\dot{X})^2. \tag{5.42}$$

We take $\epsilon^{\tau\sigma} = -\epsilon^{\sigma\tau} = +1$. In line with the analysis presented above, we take our solutions to have the property:

$$\frac{\partial X^M(\sigma,\tau)}{\partial\tau} = \mathbf{n}^M. \tag{5.43}$$

In the examples in this section, we will be using a coordinate system where $\mathbf{n}^M$ is constant. When this happens, we may solve (5.43) via (see (5.12), (5.37))

$$X^M(\sigma,\tau) = X^M(\sigma) + \mathbf{n}^M\tau. \tag{5.44}$$

As we explained above, the set of supersymmetric worldvolumes is parameterized by the set of initial shapes $X^M(\sigma)$.

On these solutions, we find

$$\sqrt{-h} = \left|X'\cdot\dot{X}\right|. \tag{5.45}$$

From the action (5.41), we can then derive the momenta

$$\begin{aligned} P_M &= \frac{\partial\mathcal{L}_{\text{brane}}}{\partial\dot{X}^M} \\ &= \frac{-e^{-\phi}}{2\pi\alpha'}\left[(G_{MN} - e^{\phi}C^{(2)}_{MN})X'^N - \mathbf{n}_M\frac{(X'\cdot X')}{X'\cdot\dot{X}}\right]. \end{aligned} \tag{5.46}$$

Since these momenta are independent of $\tau$ the equations of motion reduce to

$$-\frac{\partial \mathcal{L}_{\rm brane}}{\partial X^P} = \left( \frac{\partial (e^{-\phi} G_{MN})}{\partial X^P} + \frac{\partial C^{(2)}_{MN}}{\partial X^P} \right) (X'^M \dot{X}^N - \dot{X}^M \dot{X}^N \frac{X' \cdot X'}{X' \cdot \dot{X}}) = 0. \qquad (5.47)$$

Before we apply this general formalism to specific cases, we would like to make two comments.

1. First, as noted above, we find that $\sqrt{-h} = +|X' \cdot \dot{X}|$. If we do not put the absolute value sign, a worldsheet that folds on itself could have zero area. If we now work out the equations of motion carefully, taking into account that no such absolute value sign occurs in the coupling to the RR 2-form, then we find that unless $X' \cdot \dot{X}$ maintains a constant sign, our configurations are not solutions to the equations of motion. Here, we have taken $|X' \cdot \dot{X}| = +X' \cdot \dot{X}$. The other choice of sign, would have led to anti-branes which would not be supersymmetric in the backgrounds we consider.

2. The worldsheet may be parameterized by two coordinates, $\sigma$ and $\tau$. In many of the examples that we will consider, the vector $\mathbf{n}$ is a constant in our preferred coordinate system(see, tables 5.1 and 5.2). In such cases, we may take $t = \tau$. Now, given the profile of the string at any fixed $\tau$, we can translate each point on that profile by the integral curves of $\mathbf{n}$, to obtain the entire worldsheet. We may then use $\sigma$ to label these various integral curves of $\mathbf{n}$.

### 5.3.2 Supersymmetric Solutions in the D1-D5 background

Consider $Q_1$ D1 branes and $Q_5$ D5 branes wrapping an internal $T^4$ with sides of length $2\pi(\alpha')^{\frac{1}{2}} v^{\frac{1}{4}}$ and an $S^1$ of length $2\pi$ that we take to be along $x_5$. Table 1 describes the geometry of this background. Notice that the 3-form fluxes are normalized so that

$$\frac{1}{2\pi}\int_{S^3}\frac{G^{(3)}}{\alpha'} = 2\pi Q_5 \qquad \frac{1}{2\pi}\int_{S_3\times M_{int}}\frac{\star_{10}G^{(3)}}{\alpha'} = 2\pi Q_1. \tag{5.48}$$

If we take the near-horizon limit of the solution above, we find the geometry of $AdS_3$ in the Poincare patch, with $x_5$ identified on a circle. This is nothing but the zero mass BTZ black-hole. Although the probe solutions we present below are valid in the entire D1-D5 geometry, it will turn out that quantization of these solutions in Section (5.6) is only tractable when the probe-branes are in the near-horizon region.

The equations of motion, (5.47) reduce, on the solutions of (5.44) to

$$\frac{\partial(e^{-\phi}G_{55} + C^{(2)}_{5t})}{\partial X^P} = 0. \tag{5.49}$$

and these are manifestly satisfied since $e^{-\phi}G_{55} + C^{(2)}_{5t} = 0$.

Table 5.1 explicitly lists the solutions (5.44) and the conserved charges. The $RR$ 2-form potential in Table 5.1 has a gauge ambiguity(the coefficient $b$). The canonical momenta $P_{\phi_{1,2}}$, to begin with, depend on $b$; However, the momenta $\tilde{P}_{\phi_{1,2}}$ appearing in the both Table 5.1 and Table 5.2 (that deals with probe D-strings in global AdS) are the gauge-invariant momenta which figure in the BPS relations and do not have a gauge-ambiguity. This issue is discussed in detail in Appendix C of [85]. Note that the gauge-ambiguity is only in the magnetic part and not in case of the electric part. The reason is that it is possible to have a globally defined electric part of the potential while it is impossible to do so for the magnetic part (for reasons similar to the case of the Dirac monopole).

We now apply the general analysis presented above to obtain Table 5.1.

Table 5.1: D1-D5 system

**Geometry:**

$ds^2 = (f_1 f_5)^{-\frac{1}{2}}\left(-dt^2 + (dx_5)^2\right) + (f_1 f_5)^{\frac{1}{2}}\left(dr^2 + r^2(d\zeta^2 + \cos^2\zeta d\phi_1^2 + \sin^2\zeta d\phi_2^2)\right)$
$+\frac{e^\phi}{g} ds^2_{\text{int}}$

$e^{-2\phi} = \frac{1}{g^2}\frac{f_5}{f_1}, \quad f_1 = 1 + \frac{g\alpha' Q_1}{v r^2}, \quad f_5 = 1 + \frac{g\alpha' Q_5}{r^2}, \quad v = \frac{V}{(2\pi)^4 \alpha'^2}$

$\frac{G^{(3)}}{\alpha'} = Q_5 \sin 2\zeta d\zeta \wedge d\phi_1 \wedge d\phi_2 - \frac{2Q_1}{v f_1^2 r^3} dr \wedge dt \wedge dx_5$

$\frac{C^{(2)}}{\alpha'} = -\frac{Q_5}{2}(\cos 2\zeta + b)\zeta d\phi_1 \wedge d\phi_2 + \frac{1}{g f_1 \alpha'} dt \wedge dx_5$

**BPS Condition**

$E - L = -\int P_t d\sigma - \int P_5 d\sigma = 0$

**Null Vector tangent to worldvolume:**

$\mathbf{n}^M = \frac{\partial}{\partial t} + \frac{\partial}{\partial x_5}$

**Solution**

$t = \tau \quad x_5 = x_5(\sigma) + \tau \quad r = r(\sigma)$

$\zeta = \zeta(\sigma) \quad \phi_1 = \phi_1(\sigma) \quad \phi_2 = \phi_2(\sigma)$

$z^a_{\text{int}} = z^a_{\text{int}}(\sigma)$

**Momenta:**

$P_t = \frac{1}{2\pi\alpha' g}\left[\frac{x_5'}{f_1} - \sqrt{\frac{f_5}{f_1}}\frac{X'\cdot X'}{x_5'}\right]$

$P_5 = -\frac{1}{2\pi\alpha' g}\left[\frac{x_5'}{f_1} - \sqrt{\frac{f_5}{f_1}}\frac{X'\cdot X'}{x_5'}\right]$

$P_r = -\frac{1}{2\pi\alpha'}\left[\frac{f_5}{g} r'\right]$

$P_\zeta = -\frac{1}{2\pi\alpha'}\left[\frac{f_5 r^2 \zeta'}{g}\right]$

$\tilde{P}_{\phi_1} = -\frac{1}{2\pi\alpha'}\left[\frac{f_5 r^2 \cos^2\zeta \phi_1'}{g} + \frac{Q_5\alpha'}{2}[\cos(2\zeta) - 1]\phi_2'\right]$

$\tilde{P}_{\phi_2} = -\frac{1}{2\pi\alpha'}\left[\frac{f_5 r^2 \sin^2\zeta \phi_2'}{g} - \frac{Q_5\alpha'}{2}[\cos(2\zeta) + 1]\phi_1'\right]$

$P_{z^a} = -\frac{1}{2\pi\alpha' g}\left[g^{int}_{ab} z^{b\prime}\right]$ (internal manifold)

### 5.3.3 Supersymmetric Solutions in the D1-D5-P background

The D1-D5 system above may be generalized by adding a third charge using purely left-moving excitations which gives the 'D1-D5-P' system. The field strengths and dilaton are exactly as in Table 5.1 but the metric is altered as follows:

$$ds^2 = f_1^{-\frac{1}{2}} f_5^{-\frac{1}{2}}\left(-dt^2 + dx_5^2 + \frac{r_p^2}{r^2}(dt - dx_5)^2\right) + f_1^{\frac{1}{2}} f_5^{\frac{1}{2}}\left(dr^2 + r^2(d\zeta^2 + \cos^2\zeta d\phi_1^2 + \sin^2\zeta d\phi_2^2)\right) + \frac{e^\phi}{g} ds^2_{\text{int}} \quad (5.50)$$

Here $r_p^2 = c_p g^2 P$, where $P$ is the quantized momentum along $x_5$ and $c_p$ is a numerical constant which is not important for our purpose here.

It is easy to repeat the supersymmetry analysis above, for this background. In particular, we find that:

$$
\begin{aligned}
P_t &= \frac{1}{2\pi\alpha' g}\left[(1+\frac{r_p^2}{r^2})\frac{x_5'}{f_1} - \sqrt{\frac{f_5}{f_1}}\frac{X'\cdot X'}{x_5'}\right], \\
P_5 &= -\frac{1}{2\pi\alpha' g}\left[(1+\frac{r_p^2}{r^2})\frac{x_5'}{f_1} - \sqrt{\frac{f_5}{f_1}}\frac{X'\cdot X'}{x_5'}\right], \\
P_t &+ P_5 = 0.
\end{aligned} \tag{5.51}
$$

The rest of Table 5.1 remains valid.

### 5.3.4 Supersymmetric Solutions in the Lunin-Mathur Geometries

In this subsection, we describe supersymmetric D-string probes in the smooth 2 charge geometries of Lunin and Mathur[92, 76]. The geometry is as follows

$$
\begin{aligned}
ds^2 &= \sqrt{\frac{H}{1+K}}[-(dt - A_m dx^m)^2 + (dx_5 + B_m dx^m)^2] + \sqrt{\frac{1+K}{H}}d\vec{x}\cdot d\vec{x} \\
&\quad + \sqrt{H(1+K)}d\vec{z}\cdot d\vec{z}, \\
e^{2\phi} &= H(1+K), \quad C^{(2)}_{tm} = \frac{-B_m}{1+K}, \quad C^{(2)}_{t5} = \frac{1}{1+K}, \quad C^{(2)}_{m5} = \frac{A_m}{1+K}, \\
C^{(2)}_{mn} &= C_{mn} + \frac{A_m B_n - A_n B_m}{1+K}, \quad dB = -*dA, \quad dC = -*dH^{-1},
\end{aligned} \tag{5.52}
$$

where $H = H(\vec{x})$, $A = A(\vec{x})$ and $K = K(\vec{x})$ are three harmonic functions that are determined by 4 'string-profile' functions $F_m(v)$ as follows:

$$
\begin{aligned}
H^{-1} &= 1 + \frac{1}{2\pi}\int_0^{2\pi Q_5}\frac{dv}{|x-F(v)|^2}, \quad K = \frac{1}{2\pi}\int_0^{2\pi Q_5}\frac{|\dot{F}|^2\,dv}{|x-F(v)|^2} \\
A_m &= -\frac{1}{2\pi}\int_0^{2\pi Q_5}\frac{\dot{F}_m\,dv}{|x-F(v)|^2}.
\end{aligned} \tag{5.53}
$$

We have added 1 to $C^{(2)}_{t5}$ to be consistent with our conventions where the energy of a probe D-string infinitely far away from the parent stack of D1-D5 branes is zero. Comparing conventions with Table 5.1, we see that the parameter $g$ has been absorbed into an additive shift of the dilaton and is set to 1.

The vector $\mathbf{n} = \frac{\partial}{\partial t} + \frac{\partial}{\partial x_5}$ is null and we choose our solutions so that this vector is always tangent to the D-string worldvolume. We may apply the formalism of section 5.3.1 here to obtain

$$\begin{aligned} P_t &= -\frac{1}{2\pi\alpha'}(e^{-\phi}G_{tM} - C^{(2)}_{tM})(X^M)' - \mathbf{n}_t\gamma, \\ P_5 &= -\frac{1}{2\pi\alpha'}(e^{-\phi}G_{5M} - C^{(2)}_{5M})(X^M)' - \mathbf{n}_5\gamma, \end{aligned} \tag{5.54}$$

where we have defined $\gamma = \frac{(X')^2}{X'\cdot\dot{X}}$. We now only need to notice that $\mathbf{n}_t + \mathbf{n}_5 = 0$, $e^{-\phi}G_{55} - C^{(2)}_{t5} = 0$, $e^{-\phi}(G_{tm} + G_{5m}) + (C^{(2)}_{tm} + C^{(2)}_{5m}) = 0$ to see that the BPS condition $P_t + P_5 = 0$ is satisfied.

We comment on the relation of these geometries to global AdS in Section 5.3.5.

### 5.3.5 Supersymmetric Solutions in Global AdS

We now consider a probe D1 string propagating in global $AdS_3 \times S^3 \times M_{\text{int}}$. This geometry is described in Table 5.2. In particular, the metric is:

$$\begin{aligned} ds^2 &= G_{MN}dx^M dx^N \\ &= g\sqrt{\frac{Q_1Q_5}{v}}\alpha'\left[-\cosh^2\rho dt^2 + \sinh^2\rho d\theta^2 + d\rho^2 + d\zeta^2 + \cos^2\zeta d\phi_1^2 + \sin^2\zeta d\phi_2^2\right] \\ &+ \sqrt{\frac{Q_1}{Q_5 v}}\alpha' ds^2_{\text{int}}. \end{aligned} \tag{5.55}$$

$ds^2_{\text{int}}$ is the metric on the internal manifold. $g, v, Q_1, Q_5$ are parameters that determine the string coupling constant, volume of the internal manifold and the electric and magnetic parts of the 3-form RR field strength according to the formulae summarized in Table 5.2 below. We are following the notation of [93]. We parameterize the internal manifold using the coordinate $z^{1..4}$.

In terms of this coordinate system, the Killing spinor analysis of section 5.2.2 tells us that probe branes that preserve the Killing vector

$$\mathbf{n} = \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \phi_1} + \frac{\partial}{\partial \phi_2}$$

(i.e. branes that have $\mathbf{n}$ everywhere tangent to their world-volume) will preserve 4 of the background 16 supersymmetries.

We can now proceed as above to obtain Table 5.2

### Spectral Flow

The Global AdS geometry above corresponds to the $NS$ vacuum of the boundary CFT. The geometries considered in section 5.3.4 correspond, on the other hand to the different Ramond ground states of this CFT. Now, the NS-sector and Ramond sector in CFT with at least $(2,2)$ supersymmetry are related by an operation called spectral flow, where the Virasoro generators $L_n$ and R-symmetry current modes $J_n$ change as follows (see, e.g., [94] for a review):

$$L_n^{NS} = L_n^R + J_n^R + \frac{c}{24}\delta_{n,0}, \quad J_n^{NS} = J_n^R + \frac{c}{12}\delta_{n,0}, \tag{5.56}$$

and the moding of the fermions changes from integral to half-integral. $c$ is the central charge of the theory which, for the boundary CFT, is $6Q_1Q_5$.

Table 5.2: D branes in Global AdS

**Geometry**

$\frac{ds^2}{\alpha'} = l^2\left[-\cosh^2\rho dt^2 + \sinh^2\rho d\theta^2 + d\rho^2 + d\zeta^2 + \cos^2\zeta d\phi_1^2 + \sin^2\zeta d\phi_2^2\right] + \sqrt{\frac{Q_1}{Q_5 v}}\frac{ds^2_{\rm int}}{\alpha'}$

$e^{-2\phi} = \frac{Q_5 v}{g^2 Q_1}, l^2 = \frac{g}{\sqrt{v}}\sqrt{Q_1 Q_5}$

$\frac{G^{(3)}}{\alpha'} = \frac{*G^{(7)}}{\alpha'} = \frac{dC^{(2)}}{\alpha'} = Q_5 \sin 2\zeta d\zeta \wedge d\phi_1 \wedge d\phi_2 + Q_5 \sinh(2\rho) d\rho \wedge dt \wedge d\theta$

$\frac{C^{(2)}}{\alpha'} = -\frac{Q_5}{2}\left[(\cos 2\zeta + b) d\phi_1 \wedge d\phi_2 - (\cosh(2\rho) - 1) dt \wedge d\theta\right]$

**BPS Condition**

$E - L - J_1 - J_2 = -\int\left(P_t + P_\theta + \tilde{P}_{\phi_1} + \tilde{P}_{\phi_2}\right) d\sigma = 0$

**Null Vector tangent to worldvolume:**

$\mathbf{n}^M = \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \phi_1} + \frac{\partial}{\partial \phi_2}$

**Solution**

$t = \tau \quad \theta = \theta(\sigma) + \tau \quad \rho = \rho(\sigma)$

$\zeta = \zeta(\sigma) \quad \phi_1 = \phi_1(\sigma) + \tau \quad \phi_2 = \phi_2(\sigma) + \tau$

$z^a_{\rm int} = z^a_{\rm int}(\sigma)$

**Momenta:**

$\gamma = \frac{\sinh^2\rho\theta'^2 + \cos^2\zeta\phi_1'^2 + \sin^2\zeta\phi_2'^2 + \zeta'^2 + \rho'^2 + \frac{1}{g\alpha' Q_5} g^{\rm int}_{ab} z^{a'} z^{b'}}{\cos^2\zeta\phi_1' + \sin^2\zeta\phi_2' + \sinh^2\rho\theta'}$

$P_t = \frac{Q_5}{2\pi}\left[-\gamma\cosh^2\rho + \sinh^2\rho\theta'\right]$

$P_\theta = \frac{-Q_5}{2\pi}\left[(-\gamma + \theta')\sinh^2\rho\right]$

$\tilde{P}_{\phi_1} = \frac{-Q_5}{2\pi}\left[(-\gamma + \phi_1')\cos^2\zeta + \frac{1}{2}(\cos 2\zeta - 1)\phi_2'\right]$

$\tilde{P}_{\phi_2} = \frac{-Q_5}{2\pi}\left[(-\gamma + \phi_2')\sin^2\zeta - \frac{1}{2}(\cos 2\zeta + 1)\phi_1'\right]$

$P_\rho = \frac{-Q_5}{2\pi}\rho'$

$P_\zeta = \frac{-Q_5}{2\pi}\zeta'$

$P_{z^a} = \frac{-1}{2\pi\alpha' g}\left[g^{int}_{ab} z^{b'}\right]$ (internal manifold)

Under spectral flow, the NS vacuum maps to the Ramond vacuum with the smallest possible $U(1)$ charge of $J_0^R = -\frac{Q_1 Q_5}{2}$. It was shown in [77], that in the set of solutions (5.52), this corresponds to the profile function $F_1(v) = a\sin(wv)$, $F_2(v) = -a\cos(wv)$, $F_3(v) = F_4(v) = 0$. In our conventions, $a = \sqrt{Q_1 Q_5}, w = \frac{1}{Q_5}$. After choosing this profile function, we make the coordinate redefinitions

$$\begin{aligned} x_1 &= a\cosh\rho\sin\zeta\cos\phi_1, \quad x_2 = a\cosh\rho\sin\zeta\sin\phi_1, \\ x_3 &= a\sinh\rho\cos\zeta\cos\phi_2, \quad x_4 = a\sinh\rho\cos\zeta\sin\phi_2, \end{aligned} \tag{5.57}$$

and take the near-horizon limit(i.e drop the 1 in the harmonic functions) to obtain the metric and 3-form field strength:

$$\begin{aligned} ds^2 &= \sqrt{Q_1 Q_5}\left[-\cosh^2\rho dt^2 + \sinh^2\rho dx_5^2 + d\rho^2 + d\zeta^2\right] \\ &\quad + \sqrt{Q_1 Q_5}\left[\cos^2\zeta(d\phi_1 + dx_5)^2 + \sin^2\zeta(d\phi_2 + t)^2\right] + \sqrt{\frac{Q_1}{Q_5}}dz^i dz^i, \\ G^3 &= Q_5\sinh(2\rho)dt\wedge d\theta\wedge d\rho + Q_5\sin(2\zeta)d\zeta\wedge(d\phi_1 + dx_5)\wedge(d\phi_2 + dt). \end{aligned} \tag{5.58}$$

The dual of the 'spectral flow' (5.56) on the boundary in supergravity is the coordinate redefinition [77]

$$t_{NS} = t_R, \quad \theta_{NS} = (x_5)_R, \quad (\phi_1)_{NS} = (\phi_1)_R + (x_5)_R, \quad (\phi_2)_{NS} = (\phi_2)_R + t_R. \tag{5.59}$$

Under this mapping the solution above turns into global AdS! Moreover, going around the $\theta$ circle, once in the $NS$ sector, causes us to also go around the $(\phi_1)_{NS}$ circle to stay at constant $(\phi_1)_R$. Hence, fermions which are anti-periodic in the $NS$ sector, become periodic in the $R$ sector. One may also check that the coordinate transformation above takes:

$$\frac{\partial}{\partial t_R} + \frac{\partial}{\partial(x_5)_R} = \frac{\partial}{\partial t_{NS}} + \frac{\partial}{\partial\theta_{NS}} + \frac{\partial}{\partial(\phi_1)_{NS}} + \frac{\partial}{\partial(\phi_2)_{NS}}. \tag{5.60}$$

Thus this mapping maps the null Killing vector $\mathbf{n}$ of the Ramond sector to the special null Killing vector $\mathbf{n}$ of the NS sector. It also takes us from solutions that satisfy $E - L = 0$ to solutions that satisfy $E - L - (J_1 + J_2) = 0$.

This one to one mapping between global $AdS$ and the corresponding Lunin Mathur solution implies that everything that we say below regarding probes in global $AdS$ is also true (with appropriate redefinitions) for probes in this Lunin-Mathur geometry.

**Bound States**

The probe solutions, in global $AdS$ above have a salient feature that we wish to point out. Consider, a D-string near the boundary of $AdS$. Such a string can have finite energy only if the flux through the string almost cancels its tension. Hence, it must wrap the $\theta$ direction and we can use our freedom to redefine $\sigma$ to set $\theta' = w$. For such a string, if we take the strict $\rho \to \infty$ limit, we obtain

$$\begin{aligned} E - L &= \frac{Q_5}{2\pi}\int \gamma d\sigma \\ &= \frac{Q_5}{2\pi}\int \left[\frac{\sinh^2\rho\theta'^2 + \cos^2\zeta\phi_1'^2 + \sin^2\zeta\phi_2'^2 + \rho'^2 + G_{ab}X^{a'}X^{b'}}{\cos^2\zeta\phi_1' + \sin^2\zeta\phi_2' + \sinh^2\rho\theta'}\right] d\sigma = Q_5 w. \end{aligned} \tag{5.61}$$

Thus, we notice that for strings stretched close to the boundary, the quantity $E - L$ must be quantized in units of $Q_5$. If we wish to have intermediate values of $E-L$, our strings are 'bound' to the center of AdS. In other words the moduli space of solutions with a value of $E - L$ other than $Q_5 w$ does not include these long strings. This leads us to believe that quantum mechanically, the quantization of these solutions would lead to discrete states and not states in a continuum. This expectation is validated by the analysis of the next chapter.

The 'spectral flow' operation discussed above tells us that a similar statement holds in the geometry described by (5.58). There, what must be quantized in units of $Q_5$ is the quantity $J_1 + J_2$. On the other hand, if we consider the near-horizon of the D1-D5 geometry (see (5.121)), which is the zero mass BTZ black hole, we find that the various momenta become independent of the radial direction! This means that in that background, all probes can escape to infinity. This implies that 'averaging' over different Ramond vacua to obtain the zero mass BTZ black hole, washes out the interesting structure of 'bound-states' that we see above.

Returning now to probes in global $AdS$, those probes that do not wrap the $\theta$ direction cannot go to $\rho \to \infty$, yet their energy shows an interesting $\rho$ dependence. Consider the following solution (parameterized by $w,\ \rho_0,\ \zeta_0,\ \phi_{1_0},\ \theta_0$)

$$t = \tau, \theta(\sigma) = \theta_0,\ \rho(\sigma) = \rho_0,\ \zeta(\sigma) = \zeta_0,\ \phi_1(\sigma) = \phi_{1_0},\ \phi_2(\sigma) = w\sigma. \tag{5.62}$$

For this solution (using $w > 0$ which is necessary for supersymmetry)

$$E = Q_5 w \cosh^2(\rho_0), L = Q_5 w \sinh^2(\rho_0), P_{\phi_1} = Q_5 w, P_{\phi_2} = 0. \tag{5.63}$$

In this subsector, a given set of charges fixes $\rho_0$:

$$\sinh^2 \rho_0 = \frac{L}{wQ_5}. \tag{5.64}$$

The fact that the size of the bound state is larger for smaller $w$ is intuitively obvious; e.g. the size of an electron orbit is inversely proportional to its mass.

The equation (5.64) leads to an interesting result. The extremal BTZ black hole [95] has a horizon radius:

$$\sinh^2 \rho_h = 4MG = 4JG/l. \tag{5.65}$$

Using the values of various constants appearing in the above equation (cf. [96], p 8)

$$\begin{aligned} l \quad &= 2\pi\alpha' \sqrt{g}(Q_1 Q_5)^{1/4} V^{-1/4}, \\ G^{-1} \quad &= 2(Q_1 Q_5)^{3/4} V^{1/4}/(\pi\alpha' \sqrt{g}), \end{aligned} \tag{5.66}$$

we get for the radius of the horizon

$$\sinh^2 \rho_h = \frac{J}{Q_1 Q_5}. \tag{5.67}$$

We now make the following identifications:

| Probe configuration | BTZ |
|---|---|
| $L$ | $J$ |
| $w$ | $Q_1$ |
| $E$ | $lM+1$ |

We find that the horizon radius (5.67) exactly coincides with the size of the bound state, (5.64), under the above identifications (the third identification, of energies, follows from the second one; the extra '1' on the BTZ side owes to the mass convention used by [95] in which $\mathrm{AdS}_3$ space has mass $-1/l$).

The above agreement would appear to suggest an interpretation of the BTZ black hole as an ensemble of bound states of $Q_1$ D-string probes rotating around the center of the global $\mathrm{AdS}_3$ background at a coordinate distance $\rho_h$, given by (5.67). Since the $\mathrm{AdS}_3$ background itself is "made of" of $Q_1$ D-strings and $Q_5$ D5 branes, the above configuration is well beyond the domain of validity of the probe approximation [7] and the above interpretation should be regarded as tentative. Note that probe configurations with $w < Q_1$ have a size *larger than* the black hole radius

$$w < Q_1 \Rightarrow \rho_0 > \rho_h, \tag{5.68}$$

which, therefore, do not form a black hole.[8] The back-reacted geometry corresponding to such probe configurations is likely to be some smooth non-singular configurations. The maximum allowed value of $w(= Q_1)$ corresponds precisely to a threshold for black hole formation ($\rho_0 = \rho_h$).

[7]This is similar to the situation with $N$ dual giant gravitons in $AdS_5 \times S^5$background, at a fixed value of the global radius $\rho$.

[8]This is similar to the situation with a star, e.g. the Sun, whose size is larger than its Schwarzschild radius and hence does not form a black hole.

**Classical lower bound of energy**

It can be shown (see Appendix B of [85]) that, in global AdS, the set of solutions that we have described above has an 'energy gap'.

$$E = -\int P_t \, d\sigma \geq Q_5. \tag{5.69}$$

## 5.4 Charge Analysis: D1-D5 bound state probes

We now consider D5 branes with gauge fields on their worldvolume. Supersymmetric probes of this kind were discussed in Section 5.2.2. The embedding for such branes is given by (5.35) and the gauge fields $A_i(\sigma)$ are of the form that gives rise to (5.32)

$$F = F_{\sigma i} d\sigma \wedge dz^i + \frac{1}{2} F_{ij} dz^i \wedge dz^j. \tag{5.70}$$

with the self-duality requirement (5.33)

$$F_{ij} = \epsilon^{kl}{}_{ij} F_{kl}. \tag{5.71}$$

In this section we will obtain two results. First, we will verify the analysis of Section 5.2.2 by a charge analysis and confirm that the above configurations are indeed supersymmetric. Next, we will show that the canonical structure on the space of supersymmetric solutions of the 5+1 dimensional worldvolume theory of coincident D5 branes is identical to the canonical structure on the set of supersymmetric solutions to a 1+1 dimensional theory. For a probe comprising $p$ D1 branes and $q$ D5 branes, this effective 1+1 dimensional theory is the theory of a D-string propagating in the geometries discussed above but with the internal manifold $T^4$ or $K3$ replaced by the instanton moduli space of $p$ instantons in a $U(q)$ theory on $T^4$(or $K3$). This is similar to the result [97, 98, 79] (see, e.g. [94] for a review) that the worldvolume theory of supersymmetric D5 branes in flat space flows, in the IR, to the sigma model on the instanton moduli space. However, our result here is for D5 branes in *curved backgrounds* (discussed in Section 5.2.2) and, furthermore, the result holds (as we will see below) as long as the DBI description is valid and *we do not need to go to the IR fixed point.*

### 5.4.1 Classical Supersymmetric Bound State Solutions

We consider, first, a single D5 brane.[9]Our background has both a three form flux $G^{(3)} = dC^{(2)}$ and a seven form flux $G^{(7)} = *G^{(3)} = dC^{(6)}$. In all the examples we will consider, it is possible to define a new two-form $C'^{(2)}$ such that

$$C^{(6)} = C'^{(2)} \wedge dz^1 \wedge \ldots \wedge dz^4. \tag{5.72}$$

Using this notation, the DBI action becomes

$$\begin{aligned} S &= \int \mathcal{L} d\sigma d\tau \prod_i dz^i \\ &= -\frac{1}{(2\pi)^5\alpha'^3}\int e^{-\phi}\sqrt{-\mathrm{Det}[D_{\alpha\beta}]} + \frac{1}{(2\pi)^5\alpha'^3}\int C^{(2)} \wedge \frac{1}{2!}F \wedge F \\ &+ \frac{1}{(2\pi)^5\alpha'^3}\int C'^{(2)} \wedge dz^1 \wedge \ldots \wedge dz^4, \\ D_{\alpha\beta} &= h_{\alpha\beta} + F_{\alpha\beta}, \end{aligned} \tag{5.73}$$

where as usual $h_{\alpha\beta}$ is the pull-back of the string-frame metric to the worldvolume, $F_{\alpha\beta} = \partial_{[\alpha}A_{\beta]}$ is the two-form field strength and $A_\alpha$ is the gauge potential. It is

[9]We will be eventually interested in the instanton moduli space only for $q > 1$ D5 branes since the $q = 1$ case is rather subtle [79]. However, we include the calculations for $q = 1$ here for simplicity. The generalization to $q > 1$, which is straightforward, is left to Section 5.4.3

important to note, that we have normalized $F$ unconventionally which accounts for the absence of the usual $2\pi\alpha'$ factor. We have written the action in terms of forms to lighten the notation, but in indices: $C^{(2)} = \frac{1}{2}C^{(2)}_{MN}dX^M \wedge dX^N$.

We will now formally assume that $F$ is of the form (5.70) and write:

$$D_{\alpha\beta} = \begin{pmatrix} 0 & h_{\tau\sigma} & 0 & 0 & 0 & 0 \\ h_{\tau\sigma} & h_{\sigma\sigma} & F_{\sigma 1} & F_{\sigma 2} & F_{\sigma 3} & F_{\sigma 4} \\ 0 & -F_{\sigma 1} & e^{\phi}/g & F_{12} & F_{13} & F_{14} \\ 0 & -F_{\sigma 2} & -F_{12} & e^{\phi}/g & F_{14} & -F_{13} \\ 0 & -F_{\sigma 3} & -F_{13} & -F_{14} & e^{\phi}/g & F_{12} \\ 0 & -F_{\sigma 4} & -F_{14} & +F_{13} & -F_{12} & e^{\phi}/g \end{pmatrix}, \tag{5.74}$$

where we have assumed an internal $T^4$ with a metric $ds^2_{T^4} = \frac{e^{\phi}}{g}\sum_i dz^i dz^i$ and the embedding (5.36) or (5.37).

The Determinant of this matrix is

$$\begin{aligned} \sqrt{-|D|} &= h_{t\sigma}(\beta^2 + \frac{F_{ij}F^{ij}}{4}) \equiv h_{t\sigma}(\beta^2 + \frac{|F|^2}{2}), \\ \beta &= \frac{e^{\phi}}{g}. \end{aligned} \tag{5.75}$$

Note that:

$$|F|^2 dz^1 \wedge \ldots \wedge dz^4 = F \wedge F. \tag{5.76}$$

The field strength $F$ is derived from the gauge fields $A_i$ via $F_{\alpha\beta} = \partial_{[\alpha}A_{\beta]}$. Note that the $A_i$ have components only along the internal manifold. Let us suppose that there are solutions to (5.71) characterized by 'moduli' $\zeta^a$ (the solutions we are interested in exist, actually, for $q > 1$, so the calculations in this section and the next are to be understood in a formal sense till we apply these to $q > 1$ in Section 5.4.3).

We can assign $\sigma$ dependence to these moduli consistent with Gauss's law [90] and supersymmetry, thus

$$A_i(\sigma) = A_i(\zeta^a(\sigma)). \tag{5.77}$$

Although the moduli can vary as functions of $\sigma$, supersymmetry implies that they cannot depend on $\tau$.

To calculate the momenta, we will need the inverse of D. We have listed the relevant components of the inverse in the appendix. Using these, we find:

$$\begin{aligned}
P_M &= \frac{\delta\mathcal{L}}{\delta\dot{X}^M} \\
&= \frac{-e^{-\phi}}{(2\pi)^5\alpha'^3}\left(\sqrt{-D}\frac{D^{\tau\beta}+D^{\beta\tau}}{2}G_{MN}\partial_\beta X^N - e^{\phi}\partial_\sigma X^N\left(C^{(2)}_{MN}\frac{|F|^2}{2} + C'^{(2)}_{MN}\right)\right) \\
&= \frac{-e^{-\phi}}{(2\pi)^5\alpha'^3}\left[\left((\beta^2+\frac{|F|^2}{2})G_{MN} - \frac{e^{\phi}C^{(2)}_{MN}|F|^2}{2} - e^{\phi}C'^{(2)}_{MN}\right)\partial_\sigma X^N\right. \\
&\qquad\left. - \frac{\beta F_{\sigma i}F^i_\sigma + h_{\sigma\sigma}(\beta^2+\frac{|F|^2}{2})}{h_{\tau\sigma}}\mathbf{n}_M\right], \\
P_{Ai} &= \frac{\delta\mathcal{L}}{\delta\partial_\tau A_i} = -\frac{e^{-\phi}}{(2\pi)^5\alpha'^3}\sqrt{-D}\frac{D^{\tau i}-D^{i\tau}}{2} = \frac{e^{-\phi}\beta F_{\sigma i}}{(2\pi)^5\alpha'^3} = \frac{1}{(2\pi)^5\alpha'^3 g}\frac{\partial A_i}{\partial\zeta^\alpha}\frac{\partial\zeta^\alpha}{\partial\sigma}.
\end{aligned} \tag{5.78}$$

In the equation above, $M, N$ run over $0\ldots5$. To obtain the conserved charges of the action (5.73), we need to integrate the momenta above over all 6 worldvolume coordinates. We now proceed to show that a D5 brane that keeps the vector $\mathbf{n}^M$ of Section 5.2 tangent to its worldvolume at all points and has a worldvolume field strength of the form (5.32) is supersymmetric in the 4 backgrounds that we have discussed.

### D1-D5 background

We will discuss the D1-D5 background in some detail. The calculations required to verify supersymmetry in other backgrounds are almost identical, so we will be brief in later subsections.

In the D1-D5 background of Table 5.1

$$\begin{aligned}
\frac{G^{(3)}}{\alpha'} &= Q_5 \sin 2\zeta d\zeta \wedge d\phi_1 \wedge d\phi_2 - \frac{2Q_1}{v f_1^2 r^3} dr \wedge dt \wedge dx_5, \\
\frac{C^{(2)}}{\alpha'} &= -\frac{Q_5}{2} \cos 2\zeta d\phi_1 \wedge d\phi_2 + \frac{1}{g f_1 \alpha'} dt \wedge dx_5, \\
\frac{G^{(7)}}{\alpha'} &= \left( \frac{Q_1}{v} \sin 2\zeta d\zeta \wedge d\phi_1 \wedge d\phi_2 - \frac{2Q_5}{f_5^2 r^3} dr \wedge dt \wedge dx_5 \right) \wedge dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4, \\
\frac{C^{(6)}}{\alpha'} &= \left( \frac{-Q_1}{2v} \cos 2\zeta d\phi_1 \wedge d\phi_2 + \frac{1}{g f_5 \alpha'} dt \wedge dx_5 \right) \wedge dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4.
\end{aligned} \tag{5.79}$$

With the definition of $C'^{(2)}$ above, we have:

$$\frac{C'^{(2)}}{\alpha'} = \left( \frac{-Q_1}{2v} \cos 2\zeta d\phi_1 \wedge d\phi_2 + \frac{1}{g f_5 \alpha'} dt \wedge dx_5 \right). \tag{5.80}$$

Notice, that in the near horizon limit, we find $C'^{(2)} = \frac{e^{2\phi}}{g^2} C^{(2)}$.

To check the supersymmetry condition, we explicitly calculate $P_t$ and $P_5$ using (5.78).

$$\begin{aligned}
(2\pi)^5 \alpha'^3 P_t &= -\frac{F_{\sigma i} F^i_\sigma}{g x'_5} - \frac{e^{-\phi} h_{\sigma\sigma} (\beta^2 + \frac{|F|^2}{2})}{x'_5} - C^{(2)}_{5t} (\beta^2 + \frac{|F|^2}{2}) x'_5, \\
(2\pi)^5 \alpha'^3 P_5 &= \frac{F_{\sigma i} F^i_\sigma}{g x'_5} + \frac{e^{-\phi} h_{\sigma\sigma} (\beta^2 + \frac{|F|^2}{2})}{x'_5} - (\beta^2 + \frac{|F|^2}{2}) e^{-\phi} G_{55} x'_5,
\end{aligned} \tag{5.81}$$

where we have used that

$$C'^{(2)}_{5t} = \beta^2 C^{(2)}_{5t}. \tag{5.82}$$

Using $G_{00} = -G_{55}$ and $e^{-\phi} G_{55} + C^{(2)}_{5t} = 0$(See Table 5.1), we see that

$$E - L = \int (P_t + P_5) \; d\tau d\sigma dz^1 \dots dz^4 = 0, \tag{5.83}$$

and hence, the BPS relation is satisfied.

If we integrate (5.78) to obtain the conserved charges we see that in the near-horizon limit, where $C'^{(2)} = \frac{e^{2\phi}}{g^2}C^{(2)}$, the formulae for the energy, angular momentum and other charges are almost identical in structure to Table 5.1 except that

$$\frac{1}{2\pi\alpha'} \to \frac{1}{2\pi\alpha'}\left(\beta^2 v + \frac{1}{32\pi^4\alpha'^2}\int |F|^2 d^4 z^i\right). \tag{5.84}$$

Hence, turning on the gauge fields simply renormalizes the tension according to the 'instanton number' (5.38).[10] This equation is the precursor to the more general (5.102).

### D1-D5-P Geometry

The discussion for the D1-D5-P geometry specified by equation (5.50) is almost identical to the one above. The only modification is that we find:

$$\begin{aligned}
(2\pi)^5\alpha'^3 P_t &= -\frac{F_{\sigma i}F^i_\sigma}{gx'_5} - \frac{e^{-\phi}h_{\sigma\sigma}(\beta^2 + \frac{|F|^2}{2})}{x'_5} - \left(C^{(2)}_{5t} + e^{-\phi}G_{5t}\right)(\beta^2 + \frac{|F|^2}{2})x'_5, \\
(2\pi)^5\alpha'^3 P_5 &= \frac{F_{\sigma i}F^i_\sigma}{gx'_5} + \frac{e^{-\phi}h_{\sigma\sigma}(\beta^2 + \frac{|F|^2}{2})}{x'_5} - (\beta^2 + \frac{|F|^2}{2})e^{-\phi}G_{55}x'_5,
\end{aligned} \tag{5.85}$$

In the new background (5.50), we have $e^{-\phi}(G_{55} + G_{5t}) + C^{(2)}_{5t} = 0$. Hence, the BPS relation follows.

### Lunin-Mathur Geometries

To check the BPS condition for bound state probes in the Lunin-Mathur geometries, we need to derive an expression for $C'^{(2)}$ which is defined by (5.72). At first sight, this may seem a formidable task, but the result is quite intuitive. It may be

[10] This will become the *real* instanton number for $q > 1$ in Section 5.4.3

shown that $C'^{(2)}$ is obtained by taking $C^{(2)}$ in (5.52) and performing the substitution $H \leftrightarrow \frac{1}{1+K}$. So

$$\begin{aligned} &C'^{(2)}_{tm} = -B_m H, \quad C'^{(2)}_{t5} = H, \quad C'^{(2)}_{m5} = HA_m, \quad C'^{(2)}_{mn} = C'_{mn} + H\left(A_m B_n - A_n B_m\right), \\ &dB = - * dA, \quad dC' = - * d(1+K). \end{aligned} \tag{5.86}$$

Now, we only need to notice that $C'^{(2)}_{tM} = \beta^2 C^{(2)}_{tM}, C'(2)_{5M} = \beta^2 C^{(2)}_{5M}$, $\forall M$[11] and repeat the argument for the D1-D5 system above to see that $P_t + P_5 = 0$.

### Global AdS

The analysis, with gauge fields turned on in the D5 brane worldvolume is almost identical to the analysis in the full D1-D5 background. Here, we find

$$\frac{C'^{(2)}_{\text{global}}}{\alpha'} = \frac{e^{2\phi}}{g^2}\frac{C^{(2)}_{\text{global}}}{\alpha'} = -\frac{Q_1}{2v}\left[\cos 2\zeta d\phi_1 \wedge d\phi_2 - (\cosh(2\rho) - 1) dt \wedge d\theta\right]. \tag{5.87}$$

To check the BPS condition, let us use formula (5.78) to write down the momenta in the $t, \theta, \phi_1, \phi_2$ directions. In analogy to the analysis for the D-string, we define

$$\gamma_1 = \frac{\frac{1}{g} F_{\sigma i} F^i_\sigma + Q_5 \alpha' \left(\beta^2 + \frac{|F|^2}{2}\right)\left(\sinh^2\rho\theta'^2 + \cos^2\zeta\phi_1'^2 + \sin^2\zeta\phi_2'^2 + \zeta'^2 + \rho'^2\right)}{\cos^2\zeta\phi_1' + \sin^2\zeta\phi_2' + \sinh^2\rho\theta'}. \tag{5.88}$$

[11] As we mentioned earlier, the conventions of [77] differ slightly from [93] and $g$ has been absorbed into a shift of $\phi$. So, here $\beta = e^\phi$

with this definition, we find the momenta

$$\begin{aligned}
&(2\pi)^5\alpha'^3 P_t = -\gamma_1\cosh^2(\rho) + Q_5\alpha'\theta'\sinh^2(\rho)(\beta^2+\frac{1}{2}|F|^2),\\
&(2\pi)^5\alpha'^3 P_\theta = \gamma_1\sinh^2(\rho) - Q_5\alpha'\theta'\sinh^2(\rho)(\beta^2+\frac{1}{2}|F|^2),\\
&(2\pi)^5\alpha'^3 \tilde{P}_{\phi_1} = \gamma_1\cos^2\zeta - Q_5\alpha'\left(\beta^2+\frac{1}{2}|F|^2\right)\left(\cos^2\zeta\phi_1' - \sin^2\zeta\phi_2'\right)\phi_2',\\
&(2\pi)^5\alpha'^3 \tilde{P}_{\phi_2} = \gamma_1\sin^2\zeta + Q_5\alpha'\left(\beta^2+\frac{1}{2}|F|^2\right)\left(\cos^2\zeta\phi_1' - \sin^2\zeta\phi_2'\right)\phi_1',\\
&P_t + P_\theta + \tilde{P}_{\phi_1} + \tilde{P}_{\phi_2} = 0,
\end{aligned} \tag{5.89}$$

which verifies the BPS relation.

### 5.4.2 Obtaining an Effective Two-Dimensional Action

The space of supersymmetric solutions above, gives us a description of the supersymmetric sector of the classical phase space of the worldvolume theory defined by the action (5.73). Each solution corresponds to a point in this phase-space. Now, the action (5.73) gives rise to a canonical symplectic structure on this phase space. This structure may be encapsulated in terms of a *symplectic form*. See, for example [99] for details of this construction. We will return to this formalism again in Section 5.6. We will now show that, the classical symplectic structure on the space of supersymmetric solutions above is identical to the symplectic structure on the space of supersymmetric solutions of a 1+1 dimensional theory! This 1+1 dimensional theory will be like the theory of the D-string studied in (5.3) but propagating on a different space, where the internal manifold has been replaced by the instanton moduli space. Furthermore, we will find that the tension of this string is renormalized by a factor determined by the instanton number.

First consider the gauge fields. Recall, that in (5.78), we found that

$$p_{Ai} = \frac{1}{(2\pi)^5\alpha'^3 g}\frac{\partial A_i}{\partial\zeta^\alpha}\frac{\partial\zeta^\alpha}{\partial\sigma}. \tag{5.90}$$

The *symplectic structure* on the manifold of solutions may be written in terms of the symplectic form:

$$\Omega = \int \delta p_{Ai} \wedge \delta A_i \, d\sigma d^4 z^i, \tag{5.91}$$

where $\delta$ is an exterior derivative on the space of all solutions. $\delta A_i$ is then a 1-form in the cotangent space at the point in phase space specified by the function $A_i$ and the wedge product is taken in this cotangent space.

The $A_i$ are given as a function of the moduli $\zeta^a$ by (5.77). We can then rewrite (5.91) as:

$$\Omega = \frac{1}{(2\pi)^5\alpha'^3 g}\int \delta\left(\int d^4 z^i \frac{\partial A_i}{\partial\zeta^a}\frac{\partial A_i}{\partial\zeta^b}\zeta'^a\right) \wedge \delta\zeta^b. \tag{5.92}$$

If we define a metric on instanton moduli space,

$$g^{\rm inst}_{ab} = \frac{1}{(2\pi\sqrt{\alpha'})^4}\int d^4 z^i \frac{\partial A_i}{\partial\zeta^a}\frac{\partial A_i}{\partial\zeta^b}, \tag{5.93}$$

then, this is exactly the symplectic structure of the left-moving sector($(\zeta^a)'(\sigma,\tau) = \dot{\zeta}^a(\sigma,\tau)$) of the non-linear sigma model on the instanton moduli space defined by

$$S_{\rm inst} = \frac{1}{4\pi\alpha' g}\int g^{\rm inst}_{ab}\left(\dot{\zeta}^a\dot{\zeta}^b - (\zeta^a)'(\zeta^b)'\right) d\sigma d\tau. \tag{5.94}$$

What about the contribution of the gauge fields to the spacetime Hamiltonian? From formula (5.78) and the expressions in (5.133), we see that the gauge field momenta enter the expression for the spacetime energy only through

$$\frac{1}{(2\pi)^5\alpha'^3}\int d^4 z^i d\sigma \, \frac{F_{\sigma i}F^i_\sigma}{g} = \frac{1}{2\pi\alpha' g}\int d\sigma g^{\rm inst}_{ab}\zeta'^a\zeta'^b$$

This is exactly the Hamiltonian of the 'left-moving' sector of the non-linear sigma model (5.94).

Finally, we would like to write down an effective action that generates the symplectic structure above both in the D1-D5 system and in global AdS. To do this, first we formally extend our spacetime, by excising the coordinates on the internal manifold and including coordinates on the instanton moduli space. We now define a metric and B field on this extended space as follows:

$$\begin{aligned}
\chi^m &= \begin{pmatrix} X^M \\ \zeta^a \end{pmatrix}, \\
\mathcal{G}^1_{mn} &= \begin{pmatrix} e^{-\phi}\left(\beta^2 v + \int d^4 z^i \, \frac{|F|^2}{8\pi^2(2\pi\alpha')^2}\right) G_{MN} & 0 \\ 0 & \frac{g^{\text{inst}}_{ab}}{g} \end{pmatrix}, \\
\mathcal{B}^1 &= \left(C'^{(2)}_{MN} v + C^{(2)}_{MN} \int d^4 z^i \, \frac{|F|^2}{8\pi^2(2\pi\alpha')^2}\right) dX^M \wedge dX^N, \\
\mathcal{H}^1_{\alpha\beta} &= \mathcal{G}^1_{mn}\partial_\alpha \chi^m \partial_\beta \chi^n.
\end{aligned} \tag{5.95}$$

In the equation above, $M, N$ runs over $0 \ldots 5$, $a, b$ run over the coordinates of the instanton moduli space, $m, n$ run over both these ranges and $\alpha, \beta$ range over $\sigma, \tau$. Now, consider a sector with a fixed value of the 'instanton number' $\int d^4 z^i \frac{|F|^2}{8\pi^2(2\pi\alpha')^2}$ (see (5.38), also footnote 10). In this sector, consider the action:

$$S^1_{\text{eff}} = \frac{1}{2\pi\alpha'} \int \left(-\text{Det}[\mathcal{H}^1]\right)^{\frac{1}{2}} d\sigma d\tau + \frac{1}{2\pi\alpha'} \int \mathcal{B}^1 \tag{5.96}$$

If we look for supersymmetric solutions to the action above, we will find that they too have the property that:

$$\frac{\partial \chi^m}{\partial \tau} = \mathbf{n}^m \tag{5.97}$$

where we have extended the Killing vector field $\mathbf{n}^M$ of the previous section to this extended space in the natural way by setting its components along $\frac{\partial}{\partial\zeta^a}$ to zero. On *these* solutions, the spacetime momenta derived from the action above reproduce the momenta (5.78). Together with (5.92) this tells us the symplectic structure on supersymmetric solutions to the action (5.73) is the same as the symplectic structure on supersymmetric solutions to the action (5.96). The superscript 1 above indicates that this analysis is valid for a single D5 brane. The formula above is very suggestive and has a natural non-Abelian extension that we now proceed to discuss.

### 5.4.3 Non-Abelian Extensions

The analysis in the last two subsections was valid for a single D5 brane. It is easy to generalize the salient results to $q$ D5 branes for $q > 1$. Again, we consider a sector with fixed

$$p = \frac{1}{(2\pi\sqrt{\alpha'})^4}\int_{T^4}\frac{\mathrm{Tr}\,(F\wedge F)}{2}. \tag{5.98}$$

$p$ is now a bona-fide instanton number. In this sector consider the following natural extension to the effective quantities above given by (5.95):

$$\begin{aligned}
\chi^m &= \begin{pmatrix} X^M \\ \zeta^a \end{pmatrix}, \\
\mathcal{G}^{p,q}_{mn} &= \begin{pmatrix} e^{-\phi}\left(q\beta^2 v + p\right)G_{MN} & 0 \\ 0 & \frac{g^{\mathrm{inst}}_{ab}}{g} \end{pmatrix}, \\
\mathcal{B}^{p,q} &= \left(qC'^{(2)}_{MN}v + C^{(2)}_{MN}p\right)dX^M\wedge dX^N, \\
\mathcal{H}^{p,q}_{\alpha\beta} &= \mathcal{G}^{p,q}_{mn}\partial_\alpha\chi^m\partial_\beta\chi^n.
\end{aligned} \tag{5.99}$$

$\zeta^a$ span the moduli space of $p$ instantons in a $U(q)$ theory. We can define an effective two dimensional action for each such value of $p, q$ as:

$$S^{p,q}_{\text{eff}} = \frac{1}{2\pi\alpha'} \int (-\text{Det}[\mathcal{H}^{p,q}])^{\frac{1}{2}} + \frac{1}{2\pi\alpha'} \int \mathcal{B}^{p,q}. \tag{5.100}$$

Remarkably, we have found, that we can now apply the entire machinery of section 5.3(which we developed for D1 branes) to bound-states of D1 and D5 branes.

This result takes an especially pretty form in the near-horizon of the D1-D5 and D1-D5-P system and global AdS. Recall, that for these scenarios:

$$C'^{(2)}_{MN} = \beta^2 C^{(2)}_{MN} = \frac{Q_1}{Q_5 v} C^{(2)}_{MN}. \tag{5.101}$$

The formula (5.99) then tells us that in the near-horizon of the D1-D5 system and in global AdS(and in the corresponding Ramond sector, LM geometry), the formulae for the canonical momenta in Tables 5.1 and 5.2 are *quantitatively* correct with the following substitutions:

**1.** The internal manifold is replaced by the instanton moduli space of $p$ instantons in a $U(q)$ theory.

**2.** The tension of the 'string' is renormalized by $Q_5 \rightarrow pQ_5' + qQ_1'$. Here $Q_5'$ is the D5-charge of the background in Table 5.1 and 5.2 which must be taken to be $Q_5 - q$ in case the D5 charge of the probe is $q$ (so that the total charge at the boundary is kept fixed at $Q_5$). Similarly $Q_1' = Q_1 - p$. Thus

$$Q_5 \rightarrow p(Q_5 - q) + q(Q_1 - p) \tag{5.102}$$

## 5.5 Moving off the Special Point in Moduli Space

We can generalize the simplest D1-D5 system that we have been discussing by turning on a bulk anti self-dual $B_{NS}$ field in the background geometry.[12] This is like turning on some dissolved D3 brane charge in the background that we have taken, till now, to have only D1 and D5 charges. We should expect that the BPS solutions we have been discussing above no longer remain BPS, since a D1 or a D5 probe is not, in general, mutually supersymmetric with a D1-D3-D5 bound state (the exception is the system considered in Section 5.2.2). In this section, we will verify the expectation above by first performing a Killing spinor analysis and then by verifying our results using the DBI action.

### 5.5.1 Killing Spinor Analysis

The explicit extremal D1-D5 supergravity background with a non-zero $B_{NS}$ fields turned on was calculated in [100, 101]. We will follow [100] here. In addition to this $B_{NS}$ field and the usual 3-form RR field strength $G$, this background also has a 5-form field strength $G^{(5)}$. This solution depends on a single parameter $\varphi$ that determines the strength of the anti-self dual $B_{NS}$ field. The metric, dilaton and field strengths (adapted to our conventions regarding 'self-duality', and with $\alpha' = 1$ for simplicity)

[12] Our conventions regarding 'self-dual' and 'anti-self-dual' are the opposite of [79, 98, 100].

may be written as follows:

$$
\begin{aligned}
ds^2 &= (f_1 f_5)^{-1/2}\left[-dt^2 + dx_5^2\right] + (f_1 f_5)^{+1/2}\left(dr^2 + r^2(d\zeta^2 + \cos^2\zeta d\phi_1^2 + \sin^2\zeta d\phi_2^2)\right) \\
&\quad + (f_1 f_5)^{+1/2} Z^{-1}\left[(dx_6^2 + dx_8^2) + (dx_7^2 + dx_9^2)\right], \\
e^{2\phi} &= \frac{f_1 f_5}{Z^2}, \\
H &= dB_{\rm NS}, \\
B^{(2)}_{\rm NS} &= \left(Z^{-1}\sin(\varphi)\cos(\varphi)(f_1 - f_5) + \frac{(\mu_5 - \mu_1)\sin\varphi\cos\varphi}{\mu_5\cos^2\varphi - \mu_1\sin^2\varphi}\right)\left(dx^6\wedge dx^8 + dx^7\wedge dx^9\right), \\
G^{(3)} &= \cos^2(\varphi)\tilde{K}^{(3)} - \sin^2(\varphi)K^{(3)}, \\
G^{(5)} &= Z^{-1}\cos\varphi\sin\varphi\left(+f_5 K^{(3)} + f_1\tilde{K}^3\right)\wedge\left(dx^6\wedge dx^8 + dx^7\wedge dx^9\right),
\end{aligned}
\tag{5.103}
$$

where we defined

$$
\begin{aligned}
f_1 &= 1 + \frac{\mu_1}{r^2} \qquad f_5 = 1 + \frac{\mu_5}{r^2}, \\
\tilde{K}^{(3)} &= -\frac{f_1'}{f_1^2} dr\wedge dx^0\wedge dx_5 + \mu_5\sin(2\zeta)d\zeta\wedge d\phi_1\wedge d\phi_2, \\
K^{(3)} &= -\frac{f_5'}{f_5^2} dr\wedge dx^0\wedge dx_5 + \mu_1\sin(2\zeta)d\zeta\wedge d\phi_1\wedge d\phi_2, \\
Z &= 1 + \frac{\mu_1\sin^2(\varphi) + \mu_5\cos^2\varphi}{r^2}.
\end{aligned}
\tag{5.104}
$$

$\mu_1, \mu_5$ are parameters that determine the charges of the system according to the formulae in [100]. We alert the reader that our normalizations for $\mu_1, \mu_5$ differ from that paper by a factor of 2.

We start by calculating the bulk Killing spinors that this geometry preserves. As explained earlier the supersymmetries of the type IIB theory may be written in terms of a two-component spinor

$$
\epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}, \tag{5.105}
$$

which satisfies $\Gamma^{11}\epsilon = -\epsilon$. The dilatino Killing spinor equation is (see [102] and references therein)

$$\begin{aligned}&\left[\partial_M\phi\Gamma^M + \frac{1}{12}H_{MAB}\Gamma^{MAB}\otimes\sigma_3\right]\epsilon \\ &+\left[\frac{1}{4}e^{\phi}\sum_{n=1}^{5}\frac{(-1)^{n-1}(n-3)}{(2n-1)!}G_{A_1\ldots A_{2n-1}}\Gamma^{A_1\ldots A_{2n-1}}\otimes\lambda_n\right]\epsilon = 0,\end{aligned} \tag{5.106}$$

where $\lambda_n = \sigma_1$ for $n$ even, and $\lambda_n = i\sigma_2$ for $n$ odd. The $\{\sigma_i\}$, $i = 1, 2, 3$ are the Pauli matrices. $H$ and $G$ are the NS-NS and R-R field strengths, and $\phi$ denotes the dilaton. Our conventions are slightly different from [102] because the solution of (5.103) has $G_7 = *G_3$ and $G_5 = - * G_5$.

The spinors above are defined with respect to a particular local Lorentz frame. In our case, a convenient basis is defined by the following one-forms.

$$\begin{aligned}e^{\hat{t}} &= (f_1f_5)^{-\frac{1}{4}}dt,\\ e^{\hat{5}} &= (f_1f_5)^{-\frac{1}{4}}dx_5,\\ e^{\hat{r}} &= (f_1f_5)^{\frac{1}{4}}dr,\\ e^{\hat{\zeta}} &= (f_1f_5)^{\frac{1}{4}}rd\zeta,\\ e^{\hat{\phi}_1} &= (f_1f_5)^{\frac{1}{4}}r\cos\zeta d\phi_1,\\ e^{\hat{\phi}_2} &= (f_1f_5)^{\frac{1}{4}}r\cos\zeta d\phi_2,\\ e^{\hat{a}} &= (f_1f_5)^{\frac{1}{4}}Z^{-\frac{1}{2}}dx^a.\end{aligned} \tag{5.107}$$

Defining spinors with respect to this local Lorentz frame, we find that the Dilatino equation becomes

$$\begin{aligned}&\left[f_1^{-5/4}f_5^{-1/4}(f_1'-f_5')\Gamma^{\hat{r}}\left(\left(1-2\frac{f_1}{f_5}\frac{\sin^2(\varphi)}{\alpha}\right)\mathbb{1}-\Gamma^{\hat{0}\hat{5}}\otimes\sigma_1\right)\right]\epsilon \\ &-\left[f_5^{-5/4}f_1^{-1/4}(f_1'-f_5')\Gamma^{\hat{r}}B\left(\Gamma^{\hat{6}\hat{8}}+\Gamma^{\hat{7}\hat{9}}\right)\otimes\sigma_3\right]\epsilon = 0,\end{aligned} \tag{5.108}$$

where we defined $\alpha \equiv \cos^2(\varphi)+\frac{f_1}{f_5}\sin^2(\varphi)$, $B \equiv \sqrt{\frac{f_1}{f_5}}\frac{1}{\alpha}\sin(\varphi)\cos(\varphi) = \frac{\sqrt{f_1 f_5}\sin(\varphi)\cos(\varphi)}{f_5\cos^2(\varphi)+f_1\sin^2(\varphi)}$. All products of Gamma matrices above can be simultaneously diagonalized. We will denote the eigenvalues of $\Gamma^{\hat{0}\hat{5}}, \Gamma^{\hat{6}\hat{8}}, \Gamma^{\hat{7}\hat{9}}, \Gamma^{\hat{r}\hat{\zeta}\hat{\phi}_1\hat{\phi}_2}$ by $\pm n_1, \pm i n_2, \pm i n_3, \pm n_4$ respectively. The condition $\Gamma^{11}\epsilon = -\epsilon$ subjects these to the constraint $\prod n_1 n_2 n_3 n_4 = -1$.

Diagonalizing the matrix above is then equivalent to diagonalizing the two matrices:

$$\begin{aligned} M_1 &= n_1\sigma_1 - iB(n_2+n_3)\sigma_3, \\ M_2 &= n_4\sigma_1 + iB(n_2+n_3)\sigma_3. \end{aligned} \tag{5.109}$$

Both these matrices have eigenvalues $\pm\sqrt{1-B^2(n_2+n_3)^2}$. In particular, when $n_2 n_3 = 1 = -n_1 n_4$, there are 8 spinors that simultaneously satisfy the two equations

$$\begin{aligned} \left(\Gamma^{\hat{0}\hat{5}}\otimes\sigma_1 + B\left(\Gamma^{\hat{6}\hat{8}}+\Gamma^{\hat{7}\hat{9}}\right)\otimes\sigma_3\right)\epsilon &= \frac{f_5\cos^2\varphi - f_1\sin^2\varphi}{f_5\cos^2\varphi + f_1\sin^2\varphi}\,\epsilon, \\ \left(\Gamma^{\hat{r}\hat{\zeta}\hat{\phi}_1\hat{\phi}_2}\otimes\sigma_1 - B\left(\Gamma^{\hat{6}\hat{8}}+\Gamma^{\hat{7}\hat{9}}\right)\otimes\sigma_3\right)\epsilon &= -\frac{f_5\cos^2\varphi - f_1\sin^2\varphi}{f_5\cos^2\varphi + f_1\sin^2\varphi}\,\epsilon. \end{aligned} \tag{5.110}$$

These two equations are consistent with $\Gamma^{11}\epsilon = -\epsilon$ and satisfy the equation (5.108). They also imply $\Gamma^{6789}\epsilon = \epsilon$.

Hence, we have shown that the background defined by (5.103) preserves 8 supersymmetries that are parameterized by the projection conditions above. Notice that *none* of these spinors can be preserved by a probe D1 brane or a probe D5 brane. For arbitrary unit tangent vectors of the worldvolume $\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2$, a probe D1 brane preserves the spinors that have $\Gamma_{\hat{\mathbf{v}}_1}\Gamma_{\hat{\mathbf{v}}_2}\otimes\sigma_1\psi = \psi$. In the two dimensional space specified by (5.105) these spinors are eigenspinors of $\sigma_1$. Hence none of them coincide with the spinors that are preserved in the background above that are eigenspinors of $\sigma_1 \pm 2iB\sigma_3$.

The same argument works to show that no probe D5 branes or bound states of D1 and D5 branes can be supersymmetric in this background.

Now, consider the near-horizon limit of the geometry (5.103). In this limit, the equation above simplifies dramatically and it is easy to convince oneself that the only projection that survives above is $\Gamma^{6789}\epsilon = \epsilon$. There are 16 spinors that satisfy this equation. Hence, this is consistent with the 'doubling' of supersymmetries that is associated with the appearance of a conformal symmetry in the near-horizon limit. One may now naively suspect, that in the near-horizon a probe D-string could maintain some supersymmetries.

In the superconformal algebra, there are two types of supercharges. Conventionally, these are denoted by $Q$ – with a charge under dilatation of $+\frac{1}{2}$ – and $S$ with a dilatation charge $-\frac{1}{2}$. Now, to be BPS, we want a brane to preserve some $Q$ charges (in the superconformal algebra all primary states, whether of short representations or not are annihilated by the $S$'s). To determine which supercharges are $Q$ and which are $S$ in the near-horizon, we consider the $\hat{r}$ component of the Gravitino equation in the near-horizon limit.

The Gravitino equation reads

$$\begin{aligned}&\left[\partial_M + \frac{1}{4}w_M^{BC}\Gamma_{BC} + \frac{1}{8}H_{MAB}\Gamma^{AB}\otimes\sigma_3\right]\epsilon \\ &+\left[\frac{1}{16}e^{\phi}\sum_{n=1}^{5}\frac{(-1)^{n-1}}{(2n-1)!}G_{A_1\ldots A_{2n-1}}\Gamma^{A_1\ldots A_{2n-1}}\Gamma_M\otimes\lambda_n\right]\epsilon = 0.\end{aligned} \tag{5.111}$$

where $w_M^{BC}$ is the spin connection. In the near-horizon the $r$ component of this equation is, for the background above:

$$\frac{\partial\epsilon}{\partial r} - \frac{1}{2r}\left[\Gamma^{\hat{0}\hat{5}}\frac{(\mu_5\cos^2\varphi - \mu_1\sin^2\varphi)\sigma_1 - \sqrt{\mu_5\mu_1}\cos\varphi\sin\varphi(\Gamma^{\hat{6}\hat{8}} + \Gamma^{\hat{7}\hat{9}})\otimes(i\sigma_2)}{\mu_5\cos^2\varphi + \mu_1\sin^2\varphi}\right]\epsilon = 0. \tag{5.112}$$

If we impose $n_2 n_3 = 1$(as the dilatino equation tells us to), the square bracket on the right has eigenvalues $\pm 1$. Somewhat more remarkably, the eigenvalue $+1$ occurs when the projection condition (5.110) is satisfied. This means that the $Q$'s in the near-horizon are the same as the $Q$'s in the bulk. The new supercharges are the $S$'s. From the argument above, we now see a D-string or a D5 brane cannot be BPS even in the near-horizon. The argument for global AdS is very similar to the near-horizon argument above and instead of repeating it here, we will proceed to verify our results using a charge analysis.

### 5.5.2 Charge Analysis

In this section, we will use the DBI action to verify the results that we obtained above. For global AdS, we find the interesting result that there are still solutions to the equations of motion that preserve the Killing vector $\mathbf{n}$ but these solutions are no longer BPS.

We start by considering the extremal D1-D5 geometry. From the formulae in (5.103), we see that

$$\begin{aligned} C^{(2)}_{t5} &= \frac{f_5 \cos^2\varphi - f_1 \sin^2\varphi}{f_1 f_5}, \\ e^{-\phi} G_{55} &= \frac{Z}{f_1 f_5} = \frac{f_5 \cos^2\varphi + f_1 \sin^2\varphi}{f_1 f_5}. \end{aligned} \tag{5.113}$$

We see that the ratio between the components of the $C^{(2)}$ field and the metric has been spoilt. This effect is quite general and is the same as what we should expect if turn on a theta angle. Now, the equation of motion (5.47) for $r$ receives contributions from the following terms. (1) $X^M = x_5,\ X^N = x_5$ and (2)$X^M = x_5,\ X^N = \tau$. Since, now $e^{-\phi} G_{55} + C^{(2)}_{5t} \neq 0$, the only way to force our solutions to obey these equations is to set $(x_5)' = 0$. This confirms the expectation that, in the D1-D5 geometry, the supersymmetric brane probe solutions vanish if we move on the moduli space. It is easy to repeat the argument above to show that the same result also holds true in the D1-D5-P geometry.

The situation in global AdS is more interesting. When we take the near-horizon limit of (5.103) and translate to global coordinates, we find the metric

$$\begin{aligned} e^{-\phi} G_{MN} dx^M dx^N = Q_5' &\left( -\cosh^2\rho dt^2 + \sinh^2\rho d\theta^2 + d\rho^2 + d\zeta^2 + \cos^2\zeta d\phi_1^2 + \sin^2\zeta d\phi_2^2 \right) \\ &+ dz^i dz^i, \end{aligned} \tag{5.114}$$

and RR 2-form components

$$\begin{aligned} C^{(2)}_{\phi_1\phi_2} &= -Q_5'(1-\epsilon^2)\frac{\cos(2\zeta)}{2}, \\ C^{(2)}_{t\theta} &= Q_5'(1-\epsilon^2)\frac{\cosh(2\rho)-1}{2}, \end{aligned} \tag{5.115}$$

where

$$\begin{aligned} Q_5' &= \mu_5\cos^2\varphi + \mu_1\sin^2\varphi, \\ \epsilon^2 &= \frac{2\mu_1\sin^2\varphi}{\mu_5\cos^2\varphi + \mu_1\sin^2\varphi}. \end{aligned} \tag{5.116}$$

The equation of motion for $\rho$ now receives contributions from: $(1) X^M = \theta, X^N = \theta$ $(2) X^M = \theta, X^N = \tau$, while the equation of motion for $\zeta$ receives contributions from $(1) X^M = \phi_1, X^N = \phi_1$ $(2) X^M = \phi_2, X^N = \phi_2$ $(3) X^M = \phi_1, X^N = \phi_2$ $(4) X^M = \phi_2, X^N = \phi_1$. The identities we need are

$$\begin{aligned} e^{-\phi}G_{\theta\theta} + C^{(2)}_{\theta t} &= \epsilon^2 G_{\theta\theta} = Q_5'\epsilon^2\sinh^2\rho, \\ e^{-\phi}G_{\phi_1\phi_1} + C^{(2)}_{\phi_1\phi_2} &= \frac{Q_5'}{2}(1+\epsilon^2\cos(2\zeta)), \\ e^{-\phi}G_{\phi_2\phi_2} + C^{(2)}_{\phi_2\phi_1} &= \frac{Q_5'}{2}(1-\epsilon^2\cos(2\zeta)). \end{aligned} \tag{5.117}$$

The equations of motion are then satisfied if

$$\begin{aligned} \sinh 2\rho\theta' &= 0, \\ \sin(2\zeta)(\phi_1' - \phi_2') &= 0. \end{aligned} \tag{5.118}$$

The first equation requires us to stay at a constant point in $\theta$. The second equation requires $\phi_1' = \phi_2'$. With these constraints, one can find solutions of the form (5.44) to the equations of motion.

Unfortunately, these solutions do not maintain the BPS bound. Generalizing the formulae of table 5.2, we find that

$$\begin{aligned} P_t &= \frac{-Q_5'}{2\pi}\gamma\cosh^2\rho, \\ P_\theta &= \frac{Q_5'}{2\pi}\gamma\sinh^2\rho, \\ \tilde{P}_{\phi_1} &= \frac{Q_5'}{2\pi}\left(\gamma\cos^2\zeta - \phi_1'\cos^2\zeta - \frac{1-\epsilon^2}{2}\cos(2\zeta)\phi_2' + \frac{1-\epsilon^2}{2}\phi_2'\right), \\ \tilde{P}_{\phi_2} &= \frac{Q_5'}{2\pi}\left(\gamma\sin^2\zeta - \phi_2'\sin^2\zeta + \frac{1-\epsilon^2}{2}\cos(2\zeta)\phi_1' + \frac{1-\epsilon^2}{2}\phi_1'\right). \end{aligned} \tag{5.119}$$

Substituting, $\phi_1' = w = \phi_2'$, we find that

$$E - L - J_1 - J_2 = -\int \left(P_t + P_\theta + \tilde{P}_{\phi_1} + \tilde{P}_{\phi_2}\right)\, d\sigma = Q_5'\epsilon^2 w. \tag{5.120}$$

So, the energy of these solutions increases as we move off the special submanifold in moduli space where the anti self-dual NS-NS fluxes and theta angles are set to zero. Equation (5.120) tells us how this happens as a function of the distance in moduli space from the special submanifold.

## 5.6 Semi-Classical Quantization

The phase-space of a theory is isomorphic to the space of all its classical solutions. Using the Lagrangian, we can equip this space with a symplectic form that we can invert to calculate Dirac brackets. Then, by promoting Dirac brackets to commutators, we can use the set of classical solutions to canonically quantize the theory. The advantage of this approach is that it is covariant and that it allows us to restrict attention to special sectors of phase space by identifying the corresponding sector of classical solutions.[13] This technique has a long history and the first published reference to it, known to us, is by Dedecker [103]. Later, this was studied in [104, 105, 106, 107, 108]and then brought back into use in the eighties by [109, 99]. We refer the reader to [110] for a nice exposition of this method.

In this section, we will show how this procedure can be implemented for supersymmetric brane probes propagating in the near-horizon region of the D1-D5 system. As we explained earlier, this study has limited physical relevance because it has been argued that the extremal D1-D5 geometry is not the dual to any particular Ramond vacuum of the boundary CFT but should be thought of as an average over all Ramond vacua. In fact, even classically, we see that our probes in global AdS have the striking feature that they are generically bound the center of AdS. On quantization we would expect these to give rise to 'discrete' states. This is in sharp contrast to what we find by quantizing probes in the extremal D1-D5 background where all the states that we obtain are at the bottom of a continuum. Since, the Ramond and NS sectors of the boundary theory are related by 'spectral flow' on the boundary, this bolsters the argument above that the extremal D1-D5 geometry is only an 'average' geometry and that we should really consider probes about the geometries described in [76, 77, 78]

[13] This is valid only if the symplectic form does not mix a solution that belongs to this subset with a solution that doesn't.

Nevertheless, we include this study as an example of how these supersymmetric solutions may be quantized. A detailed study of the quantization of probes in global AdS is left to the next chapter.

Consider the near-horizon limit of the D1-D5 system. Let us define $y = \frac{\alpha' l^2}{r}$ where $l^2$ is a constant defined in the next equation. In the near horizon our background is

$$\begin{aligned} ds^2 &= l^2\alpha'\left(\frac{-dt^2+dx^2}{y^2}+\frac{dy^2}{y^2}+d\omega_3^2\right)+\sqrt{\frac{Q_1}{Q_5 v}}ds^2_{int}, \\ e^{-2\phi} &= \frac{Q_5 v}{g^2 Q_1}, \\ G^{(3)} &= Q_5\alpha'\sin(2\zeta)d\zeta\wedge d\phi_1\wedge d\phi_2 - \frac{2Q_5\alpha'}{y^3}dy\wedge dt\wedge dx_5, \\ C^{(2)} &= \frac{-Q_5\alpha'}{2}\cos 2\zeta d\phi_1\wedge d\phi_2+\frac{Q_5\alpha'}{y^2}dt\wedge dx_5, \\ l^2 &= \frac{g}{\sqrt{v}}\sqrt{Q_1Q_5}. \end{aligned} \tag{5.121}$$

The momentum conjugate to $y$ is

$$P_y = -\frac{Q_5}{2\pi}\frac{y'}{y^2}. \tag{5.122}$$

The near horizon geometry of the background described above would have been $AdS_3$ in Poincare coordinates, had the D1-branes and D5-branes not been on a circle. Adding in the circle identification, we simply get the orbifold of $AdS_3$ by a (Poincare) shift, i.e. the zero mass BTZ black hole.

Recall, from section 5.4, that we can treat all probes, D-strings or bound states of p D1 branes and q D5 branes on the same footing by performing the replacements (5.102)

$$Q_5 \to k = p(Q_5-q)+q(Q_1-p), \quad M_{\rm int}\to\mathcal{M}_{p,q}. \tag{5.123}$$

where $\mathcal{M}_{p,q}$ is the instanton moduli space of p instantons in a $SU(q)$ theory.

The symplectic form, $\Omega$ on the space of solutions is given by

$$\Omega = \int \delta P_M \wedge \delta X^M \, d\sigma, \tag{5.124}$$

where $\delta$ may be thought of as an exterior derivative in the space of solutions. Recall, the discussion in subsection 5.2.2. Apart from fixing $t = \tau$ we can use diffeomorphism invariance to set

$$x_5 = w\sigma. \tag{5.125}$$

The formula for the spacetime energy becomes

$$\begin{aligned} E &= \frac{k}{w} \int \frac{d\sigma}{2\pi} \left( \frac{y'^2}{y^2} + \cos^2 \zeta \phi_1'^2 + \sin^2 \zeta \phi_2'^2 + \zeta'^2 + \frac{g^{\text{int}}_{ab} (z^a)'(z^b)'}{kg\alpha'} \right) \\ &= \frac{E_y + E_{S^3} + E_{\text{int}}}{w}. \end{aligned} \tag{5.126}$$

Since we have fixed both $t$ and $x_5$, the $\delta P_5 \wedge \delta x_5 + \delta P_t \wedge \delta t$ terms drop out of the symplectic form, which then becomes:

$$\begin{aligned} \Omega &= \int \left( \delta P_y \wedge \delta y + \delta P_{\phi_1} \wedge \delta\phi_1 + \delta P_{\phi_2} \wedge \delta\phi_2 + \delta P_\zeta \wedge \delta\zeta + \delta P^{\text{int}}_i \wedge \delta x^i \right) \, d\sigma \\ &= \Omega_y + \Omega_{S^3} + \Omega_{\text{int}}. \end{aligned} \tag{5.127}$$

Now, if we define $y = e^\rho$, we find that

$$\begin{aligned} \delta P_y \wedge \delta y &= \frac{-k}{2\pi} \delta\rho' \wedge \delta\rho, \\ E_y &= \frac{k}{2\pi} \int (\rho')^2 d\sigma. \end{aligned} \tag{5.128}$$

We can now expand $\rho$ in modes

$$\rho = \frac{1}{\sqrt{2k|n|}} \rho_n \exp in\sigma. \tag{5.129}$$

This leads to the Dirac brackets and Hamiltonian

$$\begin{aligned} \{\rho_n, \rho_{-n}\}_{\text{D.B}} &= i, \quad n > 0 \\ E_y &= \sum_{n \in \mathbb{Z}} \frac{1}{2} n |\rho_n|^2. \end{aligned} \tag{5.130}$$

We can promote these Dirac brackets to commutators to get an infinite sequence of harmonic oscillators. We can think of these oscillators as coming from the left-moving part of a free boson. Roughly, the anti-holomorphic oscillators have been set to zero by supersymmetry. Moreover, the zero modes that tie the left and right movers together are also absent from the expression (5.130).

Now we turn to $\Omega_{S^3}$. We can map the $S^3$ into an $SU(2)$ group element using

$$g = e^{i\frac{\phi_1-\phi_2}{2}\sigma_3} e^{i\zeta\sigma_2} e^{i\frac{\phi_1+\phi_2}{2}\sigma_3}. \tag{5.131}$$

Now, introduce light-cone coordinates on the worldsheet $x^{\pm} = \tau \pm \sigma$. Consider the $WZW$ action

$$S = \frac{-k}{4\pi}\int d^2x Tr\{(g^{-1}\partial_M g)^2\} + k\Gamma_{WZ}^{\mathrm{SU(2)}}. \tag{5.132}$$

where $\Gamma_{WZ}^{\mathrm{SU(2)}}$ is the standard Wess Zumino term for the $SU(2)$ model [111]. The symplectic form and energy obtained from the action above by restricting to solutions that satisfy $\partial_+ g = 0$ coincides with $\Omega_{S^3}$ and $E_{S^3}$. Roughly speaking, we have the 'left-moving" part of the $SU(2)$ WZW model.

The quantum WZW model has a current algebra and states in its Hilbert space break up into representations of this algebra. Each representation is identified by its affine primary $[j]$ [112]. The number of affine primaries is finite and $j \in \{0, \frac{1}{2}, \ldots \frac{k}{2}\}$. What primaries occur in the spectrum above? If we consider the limit of large $k$, the WZW model describes three free bosons. If we were to quantize three bosons, $X^i(\sigma, \tau)$, using the symplectic form $\int d(X^i)' \wedge dX^i$, we would project out all right moving oscillators and all zero mode-motion. This suggests that the only affine primary in the spectrum is $[0]$.

We can obtain this result another way by using the fact that the spectrum of the

$SU(2)$ model comprises the affine primaries $\sum_{j=0}^{k/2}[j]_{\rm left} \times [j]_{\rm right}$. Since, here we have restricted the right moving-sector to be trivial, the only left-moving primary that can occur is $[0]$.

Finally, we turn to the internal degrees of freedom that correspond to fluctuations on the internal manifold. Just as above the symplectic form $\Omega_{\rm int}$ and $E_{\rm int}$ give rise to the left-moving sector of the non-linear sigma model on $\mathcal{M}_{p,q}$. We will denote this Hilbert space, which corresponds to the holomorphic part of the trivial zero mode sector of the sigma model on $\mathcal{M}_{p,q}$ by $H^0(\mathcal{M}_{p,q})$.

To conclude, we have found that the quantization of D-strings in the near-horizon of the D1-D5 system yields the left-moving part of the $R \times SU(2) \times \mathcal{M}_{p,q}$ sigma model defined on a circle of length $2\pi w$. We need to sum over all $w$ to obtain the physical spectrum.

The theory above is the Ramond sector of the theory of 'long-strings' studied in [79, 113, 114](A closely related theory was studied in [115, 116, 117]). There, it is shown how the $R \times SU(2)$ theory on the worldsheet may be embedded into a *spacetime* $N = 4$ superconformal algebra with central charge $6(k-1)$. The $N = 4$ superconformal algebra on $\mathcal{M}_{p,q}$ carries over to spacetime.

It is important to note that we do not sum over spin structures in the worldsheet theory. The fermions are always in the Ramond sector. The second important feature of the spectrum above is that it is at the bottom of a continuum of non-supersymmetric states. We can always move infitesimally away from supersymmetry by turning on the continuous momentum modes of $\rho$. This means that the Hilbert space we obtained above is of measure 'zero' in the full quantum theory.

## 5.7 Results and Discussion

In this chapter we studied brane probes in (a)the extremal D1-D5 background, (b) the extremal D1-D5-P background, (c) the smooth geometries of Lunin and Mathur with the same charges as the D1-D5 background and (d) global $AdS_3 \times S^3 \times T^4/K3$. In the first three backgrounds, states that satisfy $E-L=0$ preserve the right moving supercharges. The charge $-(E-L)$ is generated by the vector $\frac{\partial}{\partial t}+\frac{\partial}{\partial x_5}$ and we found that D-strings that maintained this vector tangent to their worldvolume at all points preserved all right moving supersymmetries. The three backgrounds above preserve 8 supersymmetries and the supersymmetric probes preserve $\frac{1}{2}$ of these. In global $AdS_3 \times S^3 \times T^4/K3$, the right moving BPS relation is $-(E-L-J_1-J_2)=0$. This combination of charges is generated by the vector $\frac{\partial}{\partial t}+\frac{\partial}{\partial \theta}+\frac{\partial}{\partial \phi_1}+\frac{\partial}{\partial \phi_2}$ and we found that D strings that keep this vector tangent to their worldvolume at all points preserve 4 right moving supersymmetries (this makes them $\frac{1}{4}$ BPS in this background). This fact allowed us to parameterize all supersymmetric D string probes in these backgrounds by their initial profiles. This result is summarized in equation (5.44).

D5 branes with self-dual gauge fields on their worldvolumes, that preserve the Killing vector above, are also supersymmetric. These gauge fields correspond to a dissolved D1 charge on the D5 worldvolume, so we interpreted supersymmetric probes of this kind as supersymmetric bound states of D1 and D5 branes. We found that these bound state probes could be described in a unified $1+1$ dimensional framework described by equations (5.99) and (5.100). This allowed us to treat them on the same footing as D1 branes.

In global $AdS$, and the corresponding Lunin-Mathur solution, the probes we found could not escape to infinity for a generic assignment of charges. This indicates that upon quantization they give rise to discrete bound states that contribute to the BPS partition function of string theory on this background. A detailed investigation of this is left to the next chapter. The fact that this structure of classical bound states is not seen in the extremal D1-D5 geometry provides further evidence for the argument that this background is not the correct dual to any Ramond vacuum in the boundary CFT.

In Section 5.5, we showed that these supersymmetric probes vanished if we turned on an anti-self-dual NS-NS field or theta angle. This means that the BPS partition function jumps as we move off the special point in moduli space where these background moduli are set to zero. This issue is discussed further in the next chapter. We note that this result is similar to the result that the $\frac{1}{8}$ and $\frac{1}{16}$ BPS partition functions of $\mathcal{N} = 4$ SYM theory on $S^3 \times R$ jump as soon as we turn on a 't Hooft coupling but are not further renormalized [14]. Finally, in section 5.6, we quantized the supersymmetric probes above in the near-horizon of the extremal D1-D5 geometry to obtain 'long-string' states at the bottom of a continuum of non-supersymmetric states.

It would be interesting to find smooth supergravity solutions that correspond to the probes above. It is possible that these solutions could be generated by using the profiles we find in the programme of [76, 77]. An ensemble of energetic spinning probes may be a useful representation of the $BTZ$ black hole. An indication of this was seen in Section 5.3. Now, in the probe approximation, we can have many probes moving in $AdS_3$ that are simultaneously supersymmetric. In global $AdS$ our analysis indicates that these probes would all be bound to $AdS$ and hence exist at a finite distance determined by their charges. If these probes have large values of $p, q$, they have many internal degrees of freedom that could give rise to a macroscopically measurable degeneracy. This suggests the interesting possibility that there may be multi-black hole solutions in global $AdS_3 \times S^3 \times T4/K3$. Similar ideas have been proposed by de Boer [118] and Sundborg [119].

## 5.A Miscellaneous Technical Details

### 5.A.1 Inverse of the Born-Infeld Matrix

The matrix D in (5.74) is simple to invert. We will only be interested in the first row and column, so we list those below:

$$\begin{aligned}
\sqrt{-|D|}D^{\tau\alpha} = \{&-\frac{\beta F_{\sigma i}F^i_\sigma + h_{\sigma\sigma}(\beta^2 + \frac{1}{2}|F|^2)}{h_{\tau\sigma}}, \beta^2 + \frac{1}{2}|F|^2, \\
&-\beta F_{\sigma 1} - F_{12}F_{\sigma 2} - F_{13}F_{\sigma 3} - F_{14}F_{\sigma 4}, F_{12}F_{\sigma 1} - \beta F_{\sigma 2} - F_{14}F_{\sigma 3} + F_{13}F_{\sigma 4}, \\
&F_{13}F_{\sigma 1} + F_{14}F_{\sigma 2} - \beta F_{\sigma 3} - F_{12}F_{\sigma 4}, F_{14}F_{\sigma 1} - F_{13}F_{\sigma 2} + F_{12}F_{\sigma 3} - \beta F_{\sigma 4}\} \\
\sqrt{-|D|}D^{\alpha\tau} = \{&-\frac{\beta F_{\sigma i}F^i_\sigma + h_{\sigma\sigma}(\beta^2 + \frac{1}{2}|F|^2)}{h_{\tau\sigma}}, \beta^2 + \frac{1}{2}|F|^2, \\
&\beta F_{\sigma 1} - F_{12}F_{\sigma 2} - F_{13}F_{\sigma 3} - F_{14}F_{\sigma 4}, F_{12}F_{\sigma 1} + \beta F_{\sigma 2} - F_{14}F_{\sigma 3} + F_{13}F_{\sigma 4}, \\
&F_{13}F_{\sigma 1} + F_{14}F_{\sigma 2} + \beta F_{\sigma 3} - F_{12}F_{\sigma 4}, F_{14}F_{\sigma 1} - F_{13}F_{\sigma 2} + F_{12}F_{\sigma 3} + \beta F_{\sigma 4}\}
\end{aligned} \tag{5.133}$$

### 5.A.2 Vielbeins

In this subsection, we list our vielbein conventions for the backgrounds considered above.

**D1-D5:**

The metric is given in Table 5.1. The Vielbein is defined by:

$$e^{\hat{t}} = (f_1 f_5)^{-\frac{1}{4}} dt, \quad e^{\hat{5}} = (f_1 f_5)^{-\frac{1}{4}} dx_5, \quad e^{\hat{r}} = (f_1 f_5)^{\frac{1}{4}} dr,$$
$$e^{\hat{\zeta}} = (f_1 f_5)^{\frac{1}{4}} r d\zeta, \quad e^{\hat{\phi}_1} = (f_1 f_5)^{\frac{1}{4}} r \cos\zeta, \quad e^{\hat{\phi}_2} = (f_1 f_5)^{\frac{1}{4}} r \sin\zeta, \quad e^a = \frac{e^{\frac{\phi}{2}}}{\sqrt{g}} dz^a. \tag{5.134}$$

**D1-D5-P:**

The metric is given in Equation 5.50. The Vielbein is defined by:

$$e^{\hat{t}} = (f_1 f_5)^{-1/4} \left( (1 - \frac{r_p^2}{r^2})^{\frac{1}{2}} dt - \frac{\frac{r_p^2}{r^2}}{\sqrt{1 - \frac{r_p^2}{r^2}}} dx_5 \right), \quad e^{\hat{5}} = (f_1 f_5)^{-1/4} (1 - \frac{r_p^2}{r^2})^{-1/2} dx_5,$$
$$e^{\hat{r}} = (f_1 f_5)^{\frac{1}{4}} dr, \quad e^{\hat{\zeta}} = (f_1 f_5)^{\frac{1}{4}} r d\zeta, \quad e^{\hat{\phi}_1} = (f_1 f_5)^{\frac{1}{4}} r \cos\zeta, \quad e^{\hat{\phi}_2} = (f_1 f_5)^{\frac{1}{4}} r \sin\zeta,$$
$$e^a = \frac{e^{\frac{\phi}{2}}}{\sqrt{g}} dz^a. \tag{5.135}$$

**Lunin-Mathur:**

The metric is given by (5.52). The Vielbein is defined by:

$$e^{\hat{t}} = \left(\frac{H}{1+K}\right)^{\frac{1}{4}} \left(dt - A_{\hat{i}} dx^{\hat{i}}\right), \quad e^{\hat{5}} = \left(\frac{H}{1+K}\right)^{\frac{1}{4}} \left(dx_5 + B_{\hat{i}} dx^{\hat{i}}\right),$$
$$e^{\hat{m}} = \left(\frac{H}{1+K}\right)^{-1/4} dx^{\hat{m}}, \quad e^{\hat{a}} = \{H(1+K)\}^{\frac{1}{4}} dx^{\hat{a}}. \tag{5.136}$$

**Global AdS:**

The metric is defined in Table 5.2. The Vielbein is defined by:

$$
\begin{aligned}
&e^{\hat{t}} = l\cosh\rho dt, \quad e^{\hat{\theta}} = l\sinh\rho d\theta, \\
&e^{\hat{\zeta}} = l d\zeta, \quad e^{\hat{\phi}_1} = l\cos\zeta, \quad e^{\hat{\phi}_2} = l\sin\zeta, \quad e^{\hat{a}} = \sqrt{\frac{Q_1}{Q_5 v}} dz^a.
\end{aligned}
\tag{5.137}
$$

# Chapter 6

# Supersymmetric States in $AdS_3/CFT_2$ II : Quantum Analysis

## 6.1 Introduction

In the previous chapter, we constructed the space of all quarter BPS brane probes in Type IIB string theory on global $AdS_3 \times S^3 \times T^4/K3$ [120]. In this chapter, we will attempt to quantize this set of solutions.

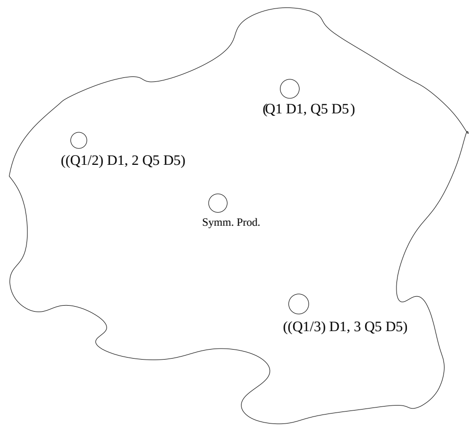
Let us briefly review the set of quantum states that we are trying to describe. The AdS/CFT conjecture relates type IIB superstring theory on global $AdS_3 \times S^3 \times T^4/K3$ to the NS sector of a (4,4) CFT living on the boundary of AdS. The NS-sector of the $N = 4$ algebra in 1+1 dimensions has short representations that are built on a special kind of lowest weight state called a chiral primary (A chiral primary has the property that its R-charge is equal to its conformal weight). $\frac{1}{4}$ BPS states of the boundary theory are of the form $|\text{anything}\rangle|\text{chiral primary}\rangle$. The probe solutions that we will discuss are dual to these states. Hence, by quantizing these solutions, we expect to obtain a description of the $\frac{1}{4}$ BPS partition function of the boundary CFT.

Now, we recall that one of the interesting features of these probes in global AdS is that, for a generic assignment of charges (i.e spacetime momenta), they are bound to the center of AdS and cannot escape to infinity. This, however, makes the quantization of these probes difficult since, in the interior of AdS, the natural symplectic structure on the space of solutions couples different degrees of freedom to each other in a complicated manner. To circumvent this difficulty, we first rewrite the supersymmetric probe solutions of Chapter 5 as left-moving classical solutions of a 'Polyakov' type non-linear sigma model. The 'bound states' above then give rise to states in discrete representations of the $SL(2, R)$ WZW model on $AdS_3$.

As we mentioned in Chapter 5, string theory on $\mathrm{AdS}_3$ has several moduli, or parameters, that we can adjust. This picture is explained further in Figure 6.1, where we have drawn a cartoon of the moduli space of D1-D5 system. At some point in the moduli space, the system may be described by the symmetric product CFT on the symmetric product $(T^4)^N/S_N$. At other points, it has a description as the low energy limit of the theory on a stack of $Q_1$ D1 and $Q_5$ D5 branes with no fluxes or theta angles, where $Q_1, Q_5$ are relatively prime divisors of $N$. Notice, that there are many possible factorizatios of $N$. There is a separate theory for each factorization, but all these worldvolume theories are believed to be continuously connected. It is possible to go from one theory to the other by tuning the $B_{\mathrm{NS}}$ fluxes and theta angles on the worldvolume. This point is discussed further in [121].

Supersymmetric probes exist only when the bulk theta angle and NS-NS fields are

Figure 6.1: Cartoon: Moduli Space of the D1-D5 system




set to zero i.e when the system is described as a 'pure' D1-D5 system. Now, it is well known that, for these parameters, the boundary theory is singular because the stack of D1 and D5 branes that make up the background can separate at no cost in energy [79]. This leads to the presence of a continuum in the spectrum that vanishes as soon as we turn on a theta angle or NS-NS fields. It is natural to ask if this continuum meets the space of $\frac{1}{4}$ BPS states. Generically, as we have explained, $\frac{1}{4}$ BPS states are described by discrete states that do not lie in a continuum. However, it turns out, that for a very special assignment of charges, supersymmetric probes in AdS can escape to infinity. Semi-classically, the quantization of these special probes gives rise to states at the bottom of continua.

So, when the theta angle and NS-NS fields are set to zero, the $\frac{1}{4}$ BPS partition function has an intricate structure, that we will describe in some detail, with unambiguous contributions from all the discrete states. As soon as we turn on one of the bulk moduli above, the $\frac{1}{4}$ BPS partition function jumps. Such a process can happen when short representations combine in pairs to form long representations. For example, in $\mathcal{N} = 4$ Yang Mills theory on $S^3 \times R$, which is dual to type IIB string theory on $AdS_5 \times S^5$ it is known that both the $\frac{1}{16}$ and $\frac{1}{8}$ BPS partition functions jump as soon as we turn on a 't Hooft coupling and are not further renormalized [14].

However, by taking appropriate limits of the $\frac{1}{4}$ BPS partition function, one may obtain two 'protected' quantities: the elliptic genus and the spectrum of $\frac{1}{2}$ BPS states (these are built on a lowest weight state of the form $|\text{chiral primary}\rangle|\text{chiral primary}\rangle$).[1] These quantities are protected in that they do not change as we move about on the moduli space unless the spectrum changes discontinuously at some point. So it is of interest to compare our results for these quantities with their known values at the point in moduli space where the boundary theory becomes a symmetric product.

Now, de Boer, building on [93], found that the low energy structure of the $\frac{1}{2}$ BPS partition function and elliptic genus of the symmetric product had a striking property [122, 123]. He found that these partition functions, for energies lower than the BTZ black hole threshold, were completely explained by gravitons and multi-gravitons,

[1] Since the terms $\frac{1}{4}$ BPS partition function and elliptic genus are, unfortunately, sometimes used interchangeably in the literature, we pause here to review our terminology. In a (4,4) theory, states may be indexed by their left and right moving conformal weights $h, \bar{h}$ and R-charges $r, \bar{r}$. The partition function depends on 4 chemical potentials:

$$Z(\beta, \bar{\beta}, \rho, \bar{\rho}) = \mathrm{Tr} e^{-\beta h - \bar{\beta}\bar{h} - \rho r - \bar{\rho}\bar{r}}. \tag{6.1}$$

The $\frac{1}{4}$ BPS partition function depends on 3 chemical potentials and is given by:

$$Z_{\frac{1}{4}}(\beta, \rho, \bar{\mu}) = \lim_{\bar{\beta}\to\infty} Z(\beta, \bar{\beta}, \rho, -\bar{\beta} + \bar{\mu}). \tag{6.2}$$

The elliptic genus depends on 2 chemical potentials and is given by:

$$E(\beta, \rho) = Z(\beta, \bar{\beta}, \rho, -\bar{\beta} + 2\pi i), \tag{6.3}$$

where the RHS is actually independent of $\bar{\beta}$. The $\frac{1}{2}$ BPS partition function also depends on 2 chemical potentials:

$$Z_{\frac{1}{2}}(\mu, \bar{\mu}) = \lim_{\beta\to\infty, -\bar{\beta}\to\infty} Z(\beta, \bar{\beta}, -\beta + \mu, -\bar{\beta} + \bar{\mu}). \tag{6.4}$$

with an appropriate exclusion principle.[2] The discussion above gives us, for the first time, a clear explanation of this phenomenon. Supersymmetric giant graviton solutions do not exist at a generic point in moduli space. Hence, almost everywhere in moduli space, the $\frac{1}{2}$ BPS partition function and elliptic genus are protected and see contributions only from gravitons and multi-gravitons at low energies.

However, we are also left with a puzzle because on this special submanifold of moduli space, giant graviton solutions *do* exist far below the black-hole threshold. Why is their signature not seen in the $\frac{1}{2}$ BPS partition function and elliptic genus evaluated at the symmetric product point? In section 6.3.4, we resolve half of this puzzle by showing that, except at very special charges, giant gravitons cannot describe $\frac{1}{2}$ BPS states! The classical solutions that correspond to generic $\frac{1}{2}$ BPS states, all the way up to the threshold of the BTZ black hole, are geodesics and not puffed up branes – gravitons rather than giant gravitons. The question of the elliptic genus is more subtle. The elliptic genus is 'blind' to right-moving charges. So, semi-classically, the sum that contributes to the elliptic genus runs not only over 'bound states' but also over the states at the bottom of continua. The presence of this continuum removes the puzzle since it invalidates the usual arguments that protect the Index.

To verify this semi-classical story, we exactly quantize the simplest of the probes above – the D-string – by dualizing to an F1-NS5 frame and using the techniques of [113, 114]. This analysis yields results that are almost entirely in accordance with our semi-classical expectations. We view this as a validation of our basic philosophy

[2]This range of energies is what contributes to the 'polar-part' of the elliptic genus. Since the elliptic genus can be almost completely reconstructed from a knowledge of its polar part [124] it appears that this Index in $AdS_3$ knows only about gravitons, just like its counterpart in $AdS_5$ [14].

that the supersymmetric sector of the full quantum theory may be understood by quantizing supersymmetric classical solutions (this idea has previously been exploited in [71, 72, 73, 74, 49, 75, 50, 51]). We find, as expected, discrete $\frac{1}{4}$ BPS states that, moreover, obey exactly the same energy formula that we obtain by semi-classical methods. By taking limits of the $\frac{1}{4}$ BPS partition function, we are also able to reproduce, almost exactly, the spectrum of $\frac{1}{2}$ BPS states of the symmetric product. However, we find, as has been found earlier [125] and as is expected from an analysis of the singularities of the boundary theory on this submanifold of moduli space [79], that some chiral-primaries are missing. These missing chiral-primaries are exactly at the point where, semi-classically, we expect to find a continuum. However, in the exact analysis, the measure for the continuum vanishes at this point. We discuss this issue further in Section 6.5.

A brief outline of this chapter is as follows. In section 6.2, we present a brief summary and review of construction of $\frac{1}{4}$ BPS brane probe solutions described in the previous chapter and discuss some toy examples of semi-classical quantization. In section 6.3, we describe a second approach to classical supersymmetric solutions that turns out to be much more convenient for purposes of quantization. In section 6.4, we discuss the quantization of these probe solutions and show that they correspond to states in discrete representations of the $SL(2, R)$ WZW model. We also describe the resultant Hilbert space in the semi-classical approximation. In Section 6.5, we perform an independent and exact quantization of the simplest of the probes above – D strings. By restricting the partition function of D-strings to its supersymmetric subsector, we validate the energy formula of section 6.4. We also use this exact calculation to discuss, more precisely, the contribution of these probes to the elliptic genus and the half-BPS partition function; the latter matches very well with the result expected from the symmetric product.

## 6.2 Classical Supersymmetric Solutions in Global $AdS_3$

### 6.2.1 Review

We start this section with a very brief, and self-contained review of the results of Chapter 5. We present only the results and no proofs.

Consider global $AdS_3 \times S^3 \times T^4$ with metric:

$$\begin{aligned} ds^2 &= G_{\mu\nu} dx^\mu dx^\nu \\ &= g\sqrt{\frac{Q_1 Q_5}{v}}\alpha' \left[-\cosh^2\rho dt^2 + \sinh^2\rho d\theta^2 + d\rho^2 + d\zeta^2 + \cos^2\zeta d\phi_1^2 + \sin^2\zeta d\phi_2^2\right] \\ &+ \sqrt{\frac{Q_1}{Q_5 v}}\alpha' ds^2_{\rm int}. \end{aligned} \tag{6.5}$$

$ds^2_{\rm int}$ is the metric on the internal $T^4$ whose sides are of length $2\pi v^{\frac{1}{4}}$ and $g, Q_1, Q_5$ are parameters that determine the string coupling constant, and the 3 form and 7 form RR field strengths according to the formulae summarized in Table 5.2 above. We are following the notation of [93]. We parameterize the internal manifold using the coordinate $z^{1\ldots4}$. Although we will concentrate here on the case where the internal manifold is $T^4$, our results may be easily generalized to $K3$.

If the theta angle (a linear combination of the RR 0 and 4 form) and NS-NS fields are set to zero then, as we explain below, this background supports $\frac{1}{4}$ BPS brane probes that consist of D1 branes, D5 branes and also bound states of $p$ D1 and $q$ D5 branes.

First, consider the case of a D-string (i.e $p = 1, q = 0$). The bosonic part of the brane action is:

$$S = \int \mathcal{L}_{\text{brane}}\, d\tau d\sigma = -\frac{1}{2\pi\alpha'}\int e^{-\phi}\sqrt{-h}\, d\tau d\sigma + \frac{1}{2\pi\alpha'}\int B_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu \frac{\epsilon^{\alpha\beta}}{2}\, d\tau d\sigma. \tag{6.6}$$

where $h = \det(G_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu)$, $\alpha, \beta$ run over the two worldsheet coordinates $\sigma, \tau$ and $B$ is the RR 2 form and $\phi$ the dilaton specified in Table 5.2. Now, consider the 'effective' metric:

$$\begin{aligned} ds^2 &= G^{\text{eff}}_{\mu\nu} dx^\mu dx^\nu \\ &= Q_5\left[-\cosh^2\rho dt^2 + \sinh^2\rho d\theta^2 + d\rho^2 + d\zeta^2 + \cos^2\zeta d\phi_1^2 + \sin^2\zeta d\phi_2^2\right] + \frac{1}{g} ds^2_{\text{int}}. \end{aligned} \tag{6.7}$$

If we define $h^{\text{eff}} = \det\left(G^{\text{eff}}_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu\right)$ then *classically* the action (5.41) may be rewritten as:

$$S = \int \mathcal{L}_{\text{brane}} = -\frac{1}{2\pi}\int \sqrt{-h^{\text{eff}}}\, d\tau d\sigma + \frac{1}{2\pi\alpha'}\int B_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu \frac{\epsilon^{\alpha\beta}}{2}\, d\tau d\sigma. \tag{6.8}$$

$Q_1$ does not appear in the effective action above and this explains why, only $Q_5$ and not $Q_1$ appears in the formulae of Table 5.2.

If we denote the energy and angular momenta in global $AdS$ by $E, L$ respectively and the two $SU(2)$ angular momenta on the $S^3$ by $\frac{J_1+J_2}{2}, \frac{J_1-J_2}{2}$ then the bulk BPS bound is

$$E - L \geq J_1 + J_2. \tag{6.9}$$

It was found in [120] that probe D-strings saturate this bound provided the vector $n^\mu = \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \phi_1} + \frac{\partial}{\partial \phi_2}$ is tangent to the brane worldvolume at all points.

Now, let us say we are given the shape of the D-string at a particular point of time. We can then translate each point on the string along the integral curves of the null vector field above to generate the entire brane worldvolume. Hence, the set of all supersymmetric brane worldvolumes is the same as the set of all initial shapes of the D-string.

The brane worldvolume is parameterized by 10 functions $X^\mu(\sigma, \tau)$. Let us now choose the coordinate $\tau$ along the brane worldvolume to be such that

$$\frac{\partial X^\mu}{\partial \tau} = n^\mu. \tag{6.10}$$

In the coordinate system of (5.55), $n^\mu$ is just a constant so we can explicitly solve the equation above for the functions $X^\mu$. We find that:

$$t = \tau,\ \theta = \theta(\sigma) + \tau,\ \rho = \rho(\sigma),\ \zeta = \zeta(\sigma),\ \phi_1 = \phi_1(\sigma) + \tau,\ \phi_2 = \phi_2(\sigma) + \tau,\ z^a = z^a(\sigma). \tag{6.11}$$

Hence, the set of all supersymmetric D-strings may be parameterized (up to a reparameterization of $\sigma$) by the set of all profile functions $\theta(\sigma), \rho(\sigma), \phi_1(\sigma), \phi_2(\sigma), z^a(\sigma)$.[3] In table 5.2 we summarize these results and also evaluate the spacetime momenta (that integrate to give conserved charges of the action (6.8)) on these solutions.

Now we turn to D5 branes. It may be shown, either by a kappa symmetry analysis or an analysis of the DBI action, that D5 branes that wrap the internal manifold and

[3] In order for the brane worldvolume to satisfy the equations of motion, it is important that the determinant of the worldsheet metric not vanish at any point. In the parameterization above, this means $X' \cdot \dot{X}$ must maintain a constant sign.

have the property that the vector $n^\mu$ is tangent to their worldvolume at each point are also supersymmetric [120]. The formulae for the momenta in Table 5.2 are then all valid but with $Q_5$ replaced by $Q_1$.

The third and last kind of supersymmetric probe is a bound state of $p$ D1 branes and $q$ D5 branes. To obtain a supersymmetric probe of this kind, we start with a stack of coincident $q$ D5 branes all of which maintain the killing vector $n^\mu$ tangent to their worldvolume at each point. Now, we turn on $U(q)$ gauge fields on the worldvolume: $A_i(\sigma)$. These are translationally invariant along $\tau$ and give rise to a field strength:

$$F = F_{\sigma i} d\sigma \wedge dz^i + \frac{1}{2} F_{ij} dz^i \wedge dz^j. \tag{6.12}$$

The condition for supersymmetry then is that this field strength be self-dual on the internal manifold: $F_{ij} = \epsilon_{ij}^{kl} F_{kl}$. We interpret this configuration as being a supersymmetric bound state of $q$ D5 branes and $p$ D1 branes, where $p$ is the instanton number of $F$

$$p = \frac{1}{8\pi^2} \int_{M_{\rm int}} \mathrm{Tr}(F \wedge F), \tag{6.13}$$

and $F$ is normalized in the conventional way. These classical instanton configurations have moduli and instead of using the gauge fields $A_i(\sigma)$ it is convenient to parameterize them in terms of their moduli $\zeta^a(\sigma)$. Note that the moduli can vary as a function of $\sigma$ without spoiling supersymmetry.

Somewhat surprisingly, the formulae for the momenta presented in Table 5.2 continue to be valid for such $(p, q)$ strings with the following two generalizations:

1. $Q_5$ is replaced by

$$k = p(Q_5 - q) + q(Q_1 - p). \tag{6.14}$$

2. the internal manifold $M_{\rm int}$ is replaced by the moduli space of p instantons in a $U(q)$ theory on $M_{\rm int}$. We will denote this manifold by $\mathcal{M}_{p,q}$. For uniformity of notation, we will henceforth use $\mathcal{M}_{1,0} \equiv M_{int}$. The coordinates $z^a$ will also be used for $\mathcal{M}_{p,q}$.

This result relies on the fact that classically, within the DBI approximation, the dynamics of the supersymmetric subsector of the 5+1 dimensional D5 brane theory reduces to the dynamics of a 1+1 dimensional sigma-model, without taking an IR limit!

The probe solutions listed above have several salient features

1. They have an energy gap – $E \geq \min\{Q_5, Q_1\}$. This is intuitive because below this energy one would expect the Hilbert space to comprise gravitons. At the minimum energy above, stringy effects in the form of these supersymmetric giant gravitons make their appearance.

2. In the AdS background that we have been discussing, we can turn on self dual NS-NS fluxes on the internal manifold and a theta angle. On the boundary, this corresponds to deforming the theory with some marginal operators [98]. The formulae of Table 5.2 are valid on the submanifold of moduli space where the coefficients of these operators are set to zero. If we move off this submanifold, there are no BPS giant graviton solutions. This means that the $\frac{1}{4}$ BPS partition function which, as we will find, has an intricate structure on this submanifold jumps as soon as we move off it. The only $\frac{1}{4}$ BPS states at a generic point in moduli space are then given by the $\frac{1}{2}$ BPS gravitons and multi-particles of these. This explains why the low energy elliptic genus and $\frac{1}{2}$ BPS partition function of

the symmetric product do not see contributions from the $\frac{1}{4}$ BPS giant gravitons that we have described.

3. Now it is well known that on the special submanifold of moduli space that we have been considering, the boundary theory is singular [79]. This raises the question as to whether the states obtained by quantizing the solutions of Table 5.2 are somehow localized about the singularities of the Higgs branch. In particular, one may worry about whether these states are located in a continuum. That this is not so, can be seen from the fact that for generic charges, these solutions are bound to the interior of AdS and cannot go off to the boundary of AdS.

   Consider a very long D-string stretched near the boundary of $AdS$. Such a string has finite energy because the flux through the string almost cancels its tension. Such a string must wrap the $\theta$ direction and we can use our residual diffeomorphism invariance to set $\theta' = w$. For such a string, if we take the strict $\rho \to \infty$ limit, we obtain

$$\begin{aligned} E - L &= \frac{Q_5}{2\pi} \int \gamma d\sigma \\ &= \frac{Q_5}{2\pi} \int \left[ \frac{\sinh^2 \rho \theta'^2 + \cos^2 \zeta \phi_1'^2 + \sin^2 \zeta \phi_2'^2 + \rho'^2 + G_{ab} X^{a'} X^{b'}}{\cos^2 \zeta \phi_1' + \sin^2 \zeta \phi_2' + \sinh^2 \rho \theta'} \right] d\sigma \\ &= Q_5 w. \end{aligned} \tag{6.15}$$

   Thus, we notice that for strings stretched close to the boundary, the quantity $E - L$ must be quantized in units of $Q_5$. For intermediate, and generic, values of $E - L$ the solutions of Table 5.2 are ‘bound’ to the center of $AdS$. This indicates that quantizing them would lead to discrete states, rather than states that are at the bottom of a continuum.

Let us elucidate point (3) above by considering another subset of solutions that do not wrap the $\theta$ circle at all. Consider the following solution (parameterized by $w,\ \rho_0,\ \zeta_0,\ \phi_{1_0},\ \theta_0$)

$$t = \tau, \quad \theta(\sigma) = \theta_0, \quad \rho(\sigma) = \rho_0, \quad \zeta(\sigma) = \zeta_0, \quad \phi_1(\sigma) = \phi_{1_0} + w\sigma, \quad \phi_2(\sigma) = w\sigma. \tag{6.16}$$

Note, that we can absorb the constant in $\phi_2(\sigma)$ into a shift in the origin of $\sigma$. For this solution (using $w > 0$ which is necessary for supersymmetry)

$$E = Q_5 w \cosh^2(\rho_0), \quad L = Q_5 w \sinh^2(\rho_0), \quad J_1 = Q_5 w \sin^2(\zeta_0), \quad J_2 = Q_5 w \cos^2(\zeta_0). \tag{6.17}$$

In this subsector, a given set of charges fixes $\rho_0$:

$$\sinh^2 \rho_0 = \frac{L}{wQ_5}. \tag{6.18}$$

Equation (5.64) has a resemblance to the formula for the size of the extremal BTZ black-hole and we refer the interested reader to [120] for the details of this analogy. This discussion provides us with an inkling of one of the main results of this chapter. Quantizing classical supersymmetric solutions in global $AdS$ generically leads to 'bound' states.

### 6.2.2 Quantization using the DBI Action: Preliminary Attempts

The space of all classical solutions of a theory is isomorphic to its phase space. The Lagrangian equips this space with a symplectic structure. This may be used to canonically quantize the theory. The advantage of this approach is that it maintains covariance. Furthermore, we can restrict attention to a subsector of phase space by identifying the corresponding classical solutions. This technique was, it seems, invented by Dedecker [103], studied in [104, 105, 106, 107, 108] and later brought back into use by [109, 99]. We refer the reader to [110] for a nice exposition of this method.

The philosophy of this chapter is that it may be possible to quantize special subsectors of solutions, for example supersymmetric subsectors, to obtain a subset of the full Hilbert space. We have enumerated all low energy supersymmetric classical solutions to Type IIB string theory on global $AdS_3$ in the previous subsection. Unfortunately it is not technically feasible to quantize all these solutions using the action (6.8) and its associated symplectic form. In section 6.4, we will show how this problem may be attacked using another method. For this subsection, however, we will restrict attention to even smaller subsectors. There is no strict justification for this since the symplectic form does couple the subset of solutions we will discuss below to other solutions not in this subset. Yet, these studies are useful as *toy examples* that yields some insight into the structure of the quantum theory.

To start with let us consider the subset of solutions (6.11) where we restrict to:

$$\theta(\sigma) = 0, \quad \rho(\sigma) = \zeta(\sigma) = 0, \quad \phi_1(\sigma) = \phi_2(\sigma) = w\sigma. \tag{6.19}$$

with an arbitrary profile on the internal manifold $\mathcal{M}_{p,q}$. For large $p, q$ this is not a severe restriction since most of the degrees of freedom of the string are in the fluctuations on $\mathcal{M}_{p,q}$.

The profile of the string on the classical instanton moduli space $\mathcal{M}_{p,q}$ is parameterized by functions $z^a(\sigma)$, with conjugate momenta $P_{z^a} = -\frac{1}{2\pi g} g^{\rm int}_{ab}(z^b)'$ where $g^{\rm int}_{ab}$ is the metric on $\mathcal{M}_{p,q}$. The spacetime energy and angular momentum are given by:

$$\frac{E+L}{2} = \frac{kw}{2} + \frac{h_{\rm int}}{w}, \qquad \frac{E-L}{2} = \frac{kw}{2}. \tag{6.20}$$

where $h_{\rm int} = \frac{1}{2\pi g}\int g^{\rm int}_{ab}(z^a)'(z^b)'d\sigma$ is the 'level' of the sigma model on $\mathcal{M}_{p,q}$.

To see what happens when we quantize the canonical structure above, consider the space of functions $X(\sigma)$ with the symplectic form:

$$\Omega = \int -\delta X'(\sigma) \wedge \delta X(\sigma)\frac{d\sigma}{2\pi}. \tag{6.21}$$

Expanding $X(\sigma) = \frac{X_n}{\sqrt{2|n|}}e^{in\sigma}$, the symplectic structure (6.21) leads to the usual Dirac bracket prescription:

$$\{X_n, X_{-n}\}_{\rm D.B} = i, \quad n > 0 \tag{6.22}$$

Promoting Dirac brackets to commutators will lead to a Fock space that has the usual left moving oscillator modes of a scalar field, but no right moving oscillators or momentum zero modes. Since these zero modes are what tie the left and right movers together, what we have here is the purely 'left-moving' part of a scalar field.

In exactly the same way, in the example above, we obtain the left-moving part of the quantum non-linear sigma model on $\mathcal{M}_{p,q}$. We will denote this Hilbert space, that comprises the trivial zero mode sector, by $H^0(\mathcal{M}_{p,q})$. The energy in $AdS$ is related to the level of this CFT by the formula (6.20).

The sigma-model on $\mathcal{M}_{p,q}$ is conformal and admits an $N = 4$ supersymmetric extension. Now, the left moving level of the boundary theory is given by $\frac{E+L}{2}$ and one may think that the superconformal algebra carries over from the worldsheet to spacetime via formula (6.20). The usual Virasoro algebra (see (6.55)) is indeed invariant under the redefinition $L'_n - \delta_{n,0}\frac{c'}{24} = \frac{1}{w}(L_{wn} - \delta_{n,0}\frac{c}{24}), c' = cw$, but now we see that the shift $\frac{kw}{2}$ does not allow us to use this prescription for equation (6.20). In Section 6.4, we will see how the $N = 4$ sigma model on $\mathcal{M}_{p,q}$ is supplemented with degrees of freedom from the 'center of mass' coordinates that shift the central charge to correctly generate this shift.

To make the example above technically tractable, we were forced to fix the 'center of mass' coordinates of the strings. We will, now, relax this assumption slightly and consider the slightly different subset of solutions where we fix to

$$\theta = w\sigma + \tau, \quad \phi_1 = (\phi_1)_0 + \tau, \quad \phi_2 = (\phi_2)_0 + \tau. \tag{6.23}$$

where $(\phi_1)_0, (\phi_2)_0$ are two real constants and $w$ an integer. $\rho, \zeta$ and the profile on $\mathcal{M}_{p,q}$ remain arbitrary. On this submanifold, we can expand out the $\rho$ and $\zeta$ in (6.11) as:

$$\rho(\sigma) = \sum_{-\infty}^{\infty} \frac{\rho_n}{\sqrt{2k|n|}} e^{in\sigma}, \quad \zeta = \sum_{-\infty}^{\infty} \frac{\zeta_n}{\sqrt{2k|n|}} e^{in\sigma}. \tag{6.24}$$

The momenta of Table 5.2, then lead to the Dirac bracket prescriptions, for $n > 0$:

$$\begin{aligned} -i\{\rho_n, \rho_{-n}\}_{\mathrm{D.B}} &= 1, \\ -i\{\zeta_n, \zeta_{-n}\}_{\mathrm{D.B}} &= 1. \end{aligned} \tag{6.25}$$

Promoting these Dirac brackets to commutators leads, as we explained above, to the left-moving sector of the Hilbert space of a free scalar field. Already, we see that the spacetime momenta do not have simple quadratic expressions in terms of the 'creation' and 'annihilation' operators above, except for

$$L = \frac{N_\rho + N_\zeta + h_{\mathrm{int}}}{w}. \tag{6.26}$$

where $N_\rho, N_\zeta$ are the levels of the $\rho$ and $\zeta$ CFT and $h_{\text{int}}$ is the level of the non-linear sigma model on $\mathcal{M}_{p,q}$.[4]

The two examples above give us some insight into the structure of the full quantum theory. For example, we see that the excitations on the internal manifold $\mathcal{M}_{p,q}$ enter the formulae for spacetime energy and angular momentum in the simple fashion specified by (6.26) and (6.20). We will obtain similar formulae in the full quantization that we perform in section 6.4.

Unfortunately, it does not seem technically possible to proceed and quantize the entire space of solutions in Table 5.2 by extending these techniques. So, we will turn, in the next section to another approach to classical solutions, using the 'Polyakov' action.

## 6.3 Another Approach to Classical Solutions: 'Polyakov' Action

Although we could quantize a limited subsector of the moduli space of supersymmetric solutions above, the symplectic form and Hamiltonian on the entire moduli space do not lend themselves to simultaneous diagonalization in any simple fashion. So, we will now present another approach to analyzing classical solutions in global AdS that will be useful for quantization.

In the action, (6.8) that governs the motion of D-string, we can introduce a world-sheet metric to get rid of $\sqrt{-h^{\text{eff}}}$. We can then fix conformal gauge and introduce

[4] It is known that in the full quantum theory, both the $\rho$ CFT and the $\zeta$ CFT develop linear dilaton terms but we cannot derive these shifts in the stress tensor from our semi-classical perspective.

light-cone coordinates $x^{\pm} = \tau \pm \sigma$ to obtain the action

$$S_{\mathcal{P}} = \frac{1}{2\pi} \int (G^{\text{eff}}_{\mu\nu} + \frac{B_{\mu\nu}}{\alpha'}) \partial_+ X^\mu \partial_- X^\nu \, dx^+ dx^- . \tag{6.27}$$

This is exactly the same as the usual transition from the Nambu-Goto to the Polyakov action (the $\mathcal{P}$ stands for Polyakov) for the F-string. We emphasize that the manipulation above is purely classical.

A classical solution of the action above is equivalent to a classical solution of the DBI action only after we impose the Virasoro constraints:

$$T(x^+) = \tilde{T}(x^-) = 0. \tag{6.28}$$

where $T$ and $\tilde{T}$ are the classical left and right moving stress tensors derived from the action (6.27).

The symplectic structure on the the set of all solutions to the action (6.27) that obey the constraints (6.28), saturate the bound (6.9) and for which the worldsheet determinant never vanishes, is identical to the symplectic structure on the set of solutions to the action (6.8) saturating the bound (6.9). As we explained in the previous section, the symplectic structure on the space of supersymmetric $(p, q)$ strings is the same as the symplectic structure on the space of supersymmetric solutions to the action (6.8) with the substitutions

$$Q_5 \to p(Q_5 - q) + q(Q_1 - p), \quad M_{\text{int}} \to \mathcal{M}_{\text{p,q}}. \tag{6.29}$$

This means that, as long as we are interested only in supersymmetric solutions we can use the action (6.27), with the substitutions (6.29) for $(p, q)$ strings also. This allows us to treat $(1, 0)$ strings on the same footing as all other $(p, q)$ strings in the discussion below.

### 6.3.1 The $SL(2,R) \times SU(2)$ WZW model: Background and Notation

The action (6.27) may be recast as an $SL(2,R) \times SU(2)$ WZW model in addition to the non-linear sigma model on the internal manifold. To see this define,

$$\begin{aligned} g_1 &= e^{i\frac{t-\theta}{2}\sigma_2} e^{\rho\sigma_3} e^{i\frac{t+\theta}{2}\sigma_2}, \\ g_2 &= e^{i\frac{\phi_1-\phi_2}{2}\sigma_3} e^{i\zeta\sigma_2} e^{i\frac{\phi_1+\phi_2}{2}\sigma_3}. \end{aligned} \tag{6.30}$$

Clearly, $g_1 \in SL(2,R)$ and $g_2 \in SU(2)$. The action (6.27), with the generalization (6.29) may be written as:

$$S = \frac{-k}{4\pi} \int \mathrm{Tr}\{(g_1^{-1}\partial_\mu g)^2 + (g_2^{-1}\partial_\mu g)^2\}\, d^2x + \Gamma_{WZ}^{SU(2)} + \Gamma_{WZ}^{SL(2,R)} + S_{\mathrm{int}}, \tag{6.31}$$

where the terms $\Gamma_{WZ}^{SU(2)}$ and $\Gamma_{WZ}^{SL(2,R)}$ are the usual Wess Zumino terms for $SU(2)$ and $SL(2,R)$ respectively (see [111] and references therein for details) and $S_{\mathrm{int}}$ is the action for the non-linear sigma model on the internal manifold $\mathcal{M}_{p,q}$. We will, sometimes, find it convenient to work with the group element

$$g = g_1 \otimes g_2, \tag{6.32}$$

where $g \in SL(2,R) \times SU(2)$.

So, apart from the non-linear sigma model on $\mathcal{M}_{p,q}$, we now have exactly a WZW model of level $k$ on $SL(2,R) \times SU(2)$. The $SU(2)$ WZW model has been studied very widely, and the $SL(2,R)$ model has attracted attention in the studies of fundamental strings propagating on $AdS_3$. In what follows, we will draw heavily on the studies of [126, 113, 114].

It is important that we wish to study the WZW model on the global cover of $SL(2,R)$. In our analysis, we will need to ensure that the string worldsheet closes in global $AdS_3$ and not just in the group parameterization (6.30). This has consequences that we will mention below.

Classical solutions of the WZW model can be decomposed into a product of a left-moving solution and a right-moving solution.

$$g(x^+, x^-) = g^+(x^+)g^-(x^-). \tag{6.33}$$

The entire solution must, of course, be periodic as a function of $\sigma$, but the two individual components only need to come back to each other up to a *monodromy*, $M \in SL(2,R) \times SU(2)$.

$$\begin{aligned} g^+(x^+ + 2\pi) &= g^+(x^+)M, \\ g^-(x^- - 2\pi) &= M^{-1}g^-(x^-). \end{aligned} \tag{6.34}$$

The decomposition of equation (6.33) is not unique. Given a classical solution $g(x^+, x^-)$, a decomposition $\{g^+(x^+), g^-(x^-)\}$, and any constant group element $U$, one obtains another decomposition of the *same* solution via $\{g^+U, U^{-1}g^-\}$. Under this $M \to U^{-1}MU$. Hence, $M$ is determined only up to conjugation. Classical solutions of the WZW model may be classified by the conjugacy class of $M$.

The quantum WZW model has a current algebra symmetry and the Hilbert space breaks up into representations of this algebra. It was shown in [127] that, at least for the case of the $SU(2)$ affine algebra, all states in a particular representation have the same monodromy eigenvalue. Conversely, as we will do, one may use the monodromy to obtain information about which states occur in the spectrum.

Our model, has six right moving and six left moving conserved currents. Three correspond to $SL(2,R)$ generators, and three correspond to $SU(2)$ generators. Explicitly, these currents are given by

$$J^a(x^+) = k\mathrm{Tr}(G^a \partial_+ g_1 g_1^{-1}), \quad \tilde{J}^a(x^-) = k\mathrm{Tr}((G^a)^* g_1^{-1}\partial_- g_1),$$
$$K^i(x^+) = k\mathrm{Tr}(\frac{-i\sigma^i}{2}\partial_+ g_2 g_2^{-1}), \quad \tilde{K}^i(x^-) = k\mathrm{Tr}(\frac{-i(\sigma^i)^*}{2} g_2^{-1}\partial_- g_2). \tag{6.35}$$

In the first line, $a$ runs over the set $\{z, +, -\}$ and we take $G^z = \frac{-i\sigma^y}{2}, G^\pm = G^x \pm iG^y = \frac{1}{2}(\sigma^z \pm i\sigma^x)$. In the second line, $i$ runs over $x, y, z$. The left and right moving stress energy tensors are given by

$$T(x^+) = \frac{1}{k}(-(J^z)^2 + (J^x)^2 + (J^y)^2 + (K^x)^2 + (K^y)^2 + (K^z)^2) + T_{\mathrm{int}}(x^+),$$
$$\tilde{T}(x^-) = \frac{1}{k}(-(\tilde{J}^z)^2 + (\tilde{J}^x)^2 + (\tilde{J}^y)^2 + (\tilde{K}^x)^2 + (\tilde{K}^y)^2 + (\tilde{K}^z)^2) + \tilde{T}_{\mathrm{int}}(x^-), \tag{6.36}$$

where $T_{\mathrm{int}}(x^+), \tilde{T}_{\mathrm{int}}(x^-)$ are the left and right moving stress energy tensors of the sigma model on the internal manifold. We will only need the property that $\int T_{\mathrm{int}}(x^+)dx^+ \geq 0, \int \tilde{T}_{\mathrm{int}}(x^-)dx^- \geq 0$.

We will find it convenient to use the modes

$$T(x^+) = \sum L_n e^{inx^+}, \quad J^i(x^+) = \sum J^i_n e^{inx^+}, \quad K^i(x^+) = \sum K^i_n e^{inx^+},$$
$$\tilde{T}(x^-) = \sum \tilde{L}_n e^{inx^-}, \quad \tilde{J}^i(x^-) = \sum \tilde{J}^i_n e^{inx^-}, \quad \tilde{K}^i(x^-) = \sum \tilde{K}^i_n e^{inx^-}. \tag{6.37}$$

The energy $E$ and angular momentum $L$ in global AdS are related to the zero modes of these currents.

$$\frac{E+L}{2} = J^z_0, \quad \frac{J_1 - J_2}{2} = K^z_0, \quad \frac{E-L}{2} = \tilde{J}^z_0, \quad \frac{J_1 + J_2}{2} = \tilde{K}^z_0. \tag{6.38}$$

Hence, the BPS bound (6.9) is saturated when:

$$\tilde{J}^z_0 = \tilde{K}^z_0. \tag{6.39}$$

### 6.3.2 Solving the Right-Moving Sector

We will now show that the supersymmetry relation (6.39) and the Virasoro constraints (6.28), are enough to solve for the entire right-moving sector of the non-linear sigma model (6.27).

First, recall that even in conformal gauge, we have the freedom to redefine $x^- \to f(x^-)$. We will choose this freedom to set

$$\tilde{J}^z(x^-) = \tilde{J}^z_0, \text{ (a constant)} \tag{6.40}$$

Let us see how this gauge may be reached. From the definition of the current, (6.35), we see that under a coordinate transformation $x^-_{\text{old}} \to x^-$:

$$\tilde{J}^z(x^-_{\text{old}}) = \frac{\partial x^-}{\partial x^-_{\text{old}}} \tilde{J}^z(x^-). \tag{6.41}$$

Hence, if we define a new coordinate by

$$\frac{\partial x^-}{\partial x^-_{\text{old}}} = \frac{\tilde{J}^z(x^-_{\text{old}})}{\tilde{J}^z_0}, \tag{6.42}$$

we will explicitly reach the gauge (6.40). Notice that (6.42) is always well-defined since to obtain a solution to the Virasoro constraints, we must have $\tilde{J}^z_0 > 0$. Second, the constant $\tilde{J}^z_0$ is automatically determined by demanding that the new coordinate have the same periodicity as the old coordinate i.e. $x^-(x^-_{\text{old}} + 2\pi) = x^-(x^-_{\text{old}}) + 2\pi$. This is reassuring, since (6.38) tells us that $\tilde{J}^z_0$ is a physical quantity; so, gauge fixing should leave it unaltered.

Now, consider the Virasoro constraint

$$\tilde{L}_0 = 0. \tag{6.43}$$

In the gauge above, this reads

$$-(\tilde{J}_0^z)^2 + k\tilde{L}_0^{\text{int}} + \sum_{n\geq 0} |\tilde{J}_n^x|^2 + |\tilde{J}_n^y|^2 + |\tilde{K}_n^z|^2 + |\tilde{K}_n^x|^2 + |\tilde{K}_n^y|^2 = 0. \tag{6.44}$$

Using relation (6.39), we find that this implies that, except for $\tilde{J}_0^z$ which is set equal to $\tilde{K}_0^z$ by (6.39) and remains an arbitrary parameter, all the other Fourier components that appear in the expression above are set to zero!

$$\begin{aligned}
\tilde{L}_0^{\text{int}} &= 0, \\
\tilde{J}_n^x = \tilde{J}_n^y &= \tilde{K}_n^x = \tilde{K}_n^y = 0, \\
\tilde{K}_{n\neq 0}^z &= \tilde{J}_{n\neq 0}^z = 0, \\
\tilde{K}_0^z &= \tilde{J}_0^z.
\end{aligned} \tag{6.45}$$

We can now solve the equations (6.35) to completely obtain the right moving sector of our theory in terms of the single arbitrary parameter $\tilde{J}_0^z$. In particular, referring to the notation of (6.32), we see that

$$\begin{aligned}
g_1(x^+, x^-) &= g_1(x^+) \exp\left\{ i\frac{\tilde{J}_0^z}{k}\sigma_2 x^- \right\}, \\
g_2(x^+, x^-) &= g_2(x^+) \exp\left\{ i\frac{\tilde{J}_0^z}{k}\sigma_3 x^- \right\}.
\end{aligned} \tag{6.46}$$

All right moving excitations on the internal manifold are also set to zero by (6.45).

Actually, these solutions are just the solutions (6.11) in a new guise. Referring to the group parameterization (6.30), we see the solutions (6.46) translate to:

$$\begin{gathered}
t(\sigma,\tau) = t(x^+) + \frac{\tilde{J}_0^z}{2k}x^-, \quad \theta(\sigma,\tau) = \theta(x^+) + \frac{\tilde{J}_0^z}{2k}x^-, \quad \phi_1(\sigma,\tau) = \phi_1(x^+) + \frac{\tilde{J}_0^z}{2k}x^-, \\
\phi_2(\sigma,\tau) = \phi_2(x^+) + \frac{\tilde{J}_0^z}{2k}x^-, \quad \rho(\sigma,\tau) = \rho(x^+), \quad \zeta(\sigma,\tau) = \zeta(x^+), \quad z^a(\sigma,\tau) = z^a(x^+).
\end{gathered} \tag{6.47}$$

Two points are worth emphasizing.

1. By solving the right-moving side of the $SL(2,R)$ and $SU(2)$ WZW models, we have also determined the monodromy of the left-moving side.

2. The monodromy of the $SU(2)$ part and the $SL(2,R)$ part are linked, since they both depend on the same parameter $\tilde{J}_0^z$.

As we mentioned, the monodromy of the solution gives us information about which representation of the current algebra we are in. The two features above then mean that at least semi-classically, once we specify the representation of the right-moving $SL(2,R)$ current algebra this determines the representation of the left-moving $SL(2,R)$ algebra and the left and right moving $SU(2)$ current algebras(the inclusion of fermions modifies this statement slightly as we discuss in Section 6.5).

$SL(2,R)$ has three types of conjugacy classes. Given $\Gamma \in SL(2,R)$, we determine its conjugacy class to be of type elliptic $((\mathrm{tr}(\Gamma))^2 < 4)$, parabolic $((\mathrm{tr}(\Gamma))^2 = 4)$ or hyperbolic $((tr(\Gamma))^2 > 4)$. We see that, generically, the monodromy of the solutions (6.46) lies in an *elliptic* conjugacy class of the group. We will find, later, that quantizing these solutions will give rise to 'short strings' in $AdS_3$. This is linked to the observation made above that unless $E - L$ is quantized in units of $k$, our strings are bound to the center of $AdS_3$. When $\tilde{J}_0^z = \frac{nk}{2}$ in equation (6.46) for some integer $n$, the monodromy of the solutions (6.46) is $\pm 1$. This kind of solution can escape to infinity and lies at the cusp of short and long strings. One may suspect that on quantization these solutions would give rise to states at the bottom of a continuum. Semi-classically, this is indeed true. The full quantum analysis in Section 6.5 raises a puzzle regarding this that we will discuss there.

### 6.3.3 Winding Sectors

Notice, that given a solution of the form (6.46) we can generate another solution using the transformation

$$g_1(x^+, x^-) \to e^{iw_1\sigma_2\frac{x^+}{2}} g_1 e^{iw_1\sigma_2\frac{x^-}{2}},$$
$$g_2(x^+, x^-) \to e^{iw_2\sigma_3\frac{x^+}{2}} g_2 e^{iw_1\sigma_3\frac{x^-}{2}}. \tag{6.48}$$

In equation (6.46), this operation takes $\tilde{J}_0^z \to \tilde{J}_0^z + \frac{kw}{2}$.

The two parameters that determine the 'spectral flow' operation above have the property that $w_1, w_2 \in \mathbb{Z}$ and $w_2 = w_1(\text{mod})2$. Notice, two important features above. First, we have to spectral flow the left moving part of the $SL(2,R)$ model by exactly the same amount as the right moving part; this is required by the periodicity of the worldsheet in global $AdS_3$ which is the global cover of $SL(2,R)$. Supersymmetry now determines that the right-moving part of the $SU(2)$ WZW model must also be spectrally flowed by $w_1$. However, periodicity on $S^3$ merely requires $w_2 = w_1(\text{mod})2$.

Second, since $\pi_1(SU(2)) = 0$, we cannot classify the solutions of the $SU(2)$ WZW model by their winding number. This is not true for $SL(2,R)$ since $\pi_1(SL(2,R)) = \mathbb{Z}$. Solutions to the $SL(2,R)$ WZW model hence break up into sectors labelled by two integers, one for the left-moving solution and the other for the right-moving solution. Since we are considering the global cover of $SL(2,R)$, closure of the worldsheet requires the two integers to be equal. So,solutions of the WZW model with target space the global cover of $SL(2,R)$ break up into sectors labelled by an single integer $w$.

### 6.3.4 $\frac{1}{2}$ BPS states

Before we conclude our discussion of classical solutions, we would like to discuss two additional issues. The first regards 'chiral, chiral primaries' in global AdS. These are half-BPS states of the $N = 4$ algebra on the boundary and are chiral primaries on the left and on the right. This means that they satisfy the BPS relations

$$\begin{aligned} E - L &= J_1 + J_2, \\ E + L &= J_1 - J_2. \end{aligned} \tag{6.49}$$

Extending the analysis of section 6.2, we conclude that probes that maintain, both

$$\begin{aligned} n_1 &= \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \phi_1} + \frac{\partial}{\partial \phi_2}, \\ n_2 &= \frac{\partial}{\partial t} - \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \phi_1} - \frac{\partial}{\partial \phi_2}, \end{aligned} \tag{6.50}$$

preserve the required 8 supersymmetries.

From our list of solutions, we can see that this fixes both the $\sigma$ dependence and the $\tau$ dependence. In particular, the only allowed solutions are:

$$\begin{aligned} &t = \tau, \quad \theta = w\sigma + \tau, \quad \phi_1 = \text{const} + \tau, \ \phi_2 = w\sigma + \tau, \\ &\rho = \text{const}, \quad \zeta = \text{const}, \quad z^i = \text{const}. \end{aligned} \tag{6.51}$$

The two tangent vectors above are then, $\frac{\partial}{\partial \tau}$ and $\frac{\partial}{\partial \tau} - \frac{2}{w}\frac{\partial}{\partial \sigma}$.

We now encounter a surprise. Calculating the charges of these solutions from table 5.2, we find that for a $(p, q)$ probe, $E = J_1 = kw$ and $L = J_2 = 0$ where $k$ is given by (6.14). However, the boundary theory has chiral-chiral primaries for all half-integer values of the scaling dimension($\frac{E+L}{2}$) upto $\frac{Q_1 Q_5}{2}$, not just the small subset above. Evidently, smooth giant gravitons cannot describe generic chiral primaries; a point that was stressed in [77].

Moving now to the 'Polyakov' approach, we can find the classical solutions for chiral-chiral primaries by merely repeating the analysis above for the left-moving side. We find that the solutions that obey the relations (6.49) and have the correct periodicity on the worldsheet are:

$$\begin{aligned} g_1(x^+,x^-) &= \exp\left\{i\frac{J_0^z}{k}\sigma_2 x^+\right\}\exp\left\{i\frac{J_0^z}{k}\sigma_2 x^-\right\}, \\ g_2(x^+,x^-) &= \exp\left\{i\frac{J_0^z}{k}\sigma_3 x^+\right\}\exp\left\{i\frac{J_0^z}{k}\sigma_3 x^-\right\}, \end{aligned} \tag{6.52}$$

where, as before, $g_1$ is an element of $SL(2,R)$ and $g_2$ an element of $SU(2)$. The form (6.52) is unique up to an irrelevant additive shift in $\sigma$. If we move to spacetime, using the parameterization (6.30), we find that the solutions (6.52) correspond to curves that pass through $\rho = 0$ (the center of $AdS$) and sit at $\zeta = 0$.[5] Hence, the values of $\theta$ and $\phi_2$ are ill defined but if we take these values to be zero, then the solutions (6.52) correspond to $t = \frac{2J_0^z\tau}{k},\ \theta = 0,\ \rho = 0,\ \phi_1 = \frac{2J_0^z\tau}{k},\ \phi_2 = 0,\ \zeta = 0.$

We now notice a remarkable feature about these solutions. Spectral flow does not puff these geodesics into strings! The transformation (6.48) takes $J_0^z \to J_0^z + \frac{wk}{2}$ but leaves the solution in the form of a geodesic placed at $\rho = 0, \zeta = 0$. This simple observation explains several facts about the bulk spectrum of chiral-chiral primaries that have hitherto been puzzles:

1. The spectrum of chiral-chiral operators in non-zero winding sectors was calculated in a nice paper by Argurio, Giveon and Shomer [125] (AGS) and found to be a continuation of a graviton spectrum. While one may expect stringy effects to start showing at energies of order $Q_5$ or $Q_1$, this does not happen for

[5]The Polyakov formalism can accommodate these solutions because solutions to the action (6.27) obeying the constraints (6.28) comprise all solutions to the action (6.8) plus geodesics

chiral-chiral operators because spectral flow does not puff these geodesics up into strings. This also explains why de Boer, in [122], was successful in reproducing the spectrum of (1/2) BPS states on the boundary up to energies $\frac{Q_1 Q_5}{2}$ by naively extending the graviton spectrum.

2. AGS conjectured that in each winding sector, some chiral-operators (with charges integrally quantized in units of $\frac{Q_5}{2}$) vanished into the continuum. We see, that at exactly these values of the charge, chiral-chiral primaries are described by giant gravitons as in equation (6.51). Classically, they can be at any value of $\rho$ including $\rho \to \infty$. Quantum mechanically this means that they are at the bottom of a continuum of non-supersymmetric states and we may expect difficulty in counting them.

3. From the spectrum of chiral operators, AGS also discussed the possibility that the boundary theory was a deformation of the iterated symmetric product $((M_{int})^{Q_5}/S_{Q_5})^{Q_1}/S_{Q_1}$. However, since the classical solutions corresponding to chiral operators are geodesics they do not differentiate between different probes; we cannot determine if they are constituted by D1 branes, D5 branes or a bound state of these. Even in the semi-classical quantum analysis below we find that chiral-chiral operators can be obtained by quantizing any of these probes.This restores the democracy between $Q_1$ and $Q_5$.

4. Correlation functions of chiral-chiral operators, in the zero-winding sector, were recently calculated by Gaberdiel and Kirsch [128] and Dabholkar and Pakman [129]. The insight above, that chiral-chiral operators do not 'see' winding, in-

dicates that similar results would be obtained by repeating this calculation in sectors of non-zero winding.

## 6.4 Semi-Classical Quantization

We will now use the insights of the previous sections to deduce features of the $\frac{1}{4}$ BPS sector of quantum string theory on $AdS_3$. Throughout this section, we will work in a semi-classical limit, where the charges of the states that we consider are large and hence, for example $j(j+1)$ may be well approximated by $j^2$. There are two reasons for doing this. The first is that the analysis we perform here is valid for general $(p, q)$ probes. The second is we will find that when we perform an exact analysis of the D-string by dualizing to a F1-NS5 frame, it will turn out the formulae we derive in this section are *quantitatively* correct including all the additive factors of 1. The factors that we neglect, conspire to cancel!

As we mentioned in the previous section, a $(p, q)$ probe leads to an $SL(2, R) \times SU(2)$ model with level $k$ given by equation (6.14). The details of the internal manifold, $\mathcal{M}_{p,q}$ will not be too important for us here.

The approach we will adopt is as follows. We start by reviewing the Hilbert space of the $SL(2, R)$ and $SU(2)$ WZW models. The question that faces us, then is to understand what sector of this Hilbert space corresponds to the solutions (6.46) obeying the Virasoro constraints. We tackle this question in subsection 6.4.2.

### 6.4.1 The SU(2) and SL(2,R) WZW models: a review

The $SL(2,R)$ and $SU(2)$ WZW models each have 3 left moving conserved currents defined in equation (6.35) which we call $J^i$ and $K^i$ respectively. In the quantum theory these currents give rise to an affine symmetry via the commutation relations:

$$\begin{aligned}
&[J^z_n, J^\pm_m] = \pm J^\pm_{n+m}, \quad [J^z_n, J^z_m] = -\frac{kn}{2}\delta_{n+m,0}, \quad [J^+_n, J^-_m] = -2J^z_{n+m} + kn\delta_{n+m,0}.\\
&[K^z_n, K^\pm_m] = \pm K^\pm_{n+m}, \quad [K^z_n, K^z_m] = \frac{kn}{2}\delta_{n+m,0}, \quad [K^+_n, K^-_m] = 2K^z_{n+m} + kn\delta_{n+m,0}.
\end{aligned} \tag{6.53}$$

The Stress Energy tensors, for each algebra, are given by the usual Sugawara construction. In particular, the modes $L_n$ are given by:

$$\begin{aligned}
L^{SL(2,R)}_n &= \frac{1}{2(k-2)} : \left[\sum_{m=-\infty}^{+\infty} J^+_m J^-_{n-m} + J^-_m J^+_{n-m} - 2J^z_m J^z_{-m}\right] :\\
L^{SU(2)}_n &= \frac{1}{2(k+2)} : \left[\sum_{m=-\infty}^{+\infty} K^+_m K^-_{n-m} + K^-_m K^+_{n-m} + 2K^z_m K^z_{-m}\right] :
\end{aligned} \tag{6.54}$$

where $: \ldots :$ implies normal ordering where negatively moded operators are placed before positively moded operators. These modes obey the algebra:

$$\begin{aligned}
[L^{SL(2,R)}_n, J^a_m] &= -mJ^a_{n+m}, \ [L^{SU(2)}_n, K^a_m] = -mK^a_{n+m},\\
[L^{SL(2,R)}_n, L^{SL(2,R)}_m] &= (n-m)L^{SL(2,R)}_{n+m} + \frac{k}{4(k-2)}(n^3-n)\delta_{n+m,0}\\
[L^{SU(2)}_n, L^{SU(2)}_m] &= (n-m)L^{SU(2)}_{n+m} + \frac{k}{4(k+2)}(n^3-n)\delta_{n+m,0}
\end{aligned} \tag{6.55}$$

Representations of the $SU(2)$ affine algebra are constructed by starting with a lowest weight state $|j\rangle$ obeying $K^{\pm,z}_{n>0}|j\rangle = 0, \ \ K^+_0|j\rangle = 0, \ \ K^z|j\rangle = j|j\rangle$. Such a state is called an 'affine primary'. Given an affine primary, one acts in all possible ways with the lowering operators $K^{\pm,z}_{n<0}, K^-_0$ and removes null states to construct the entire representation. We will denote a representation built on an affine primary of weight

$j$ as $\mathcal{L}^j$ The spectrum of the $SU(2)$ model at level $k$ comprises the 'diagonal modular invariant' $\oplus_{j=0,\frac{1}{2}\cdots\frac{k}{2}}\mathcal{L}^j\otimes\bar{\mathcal{L}}^j$, where the left moving affine primary has the same weight as the right moving affine primary. We refer the reader to [112] for details.

The spectrum of the $SL(2,R)$ WZW model is more intricate, because this group is non-compact. The $SL(2,R)$ WZW model also has lowest weight representations of the kind described above. These are discussed in [130, 113] and we refer the interested reader there for details. Here, we review the two kinds of representations that are most relevant to strings propagating on $AdS_3$.

1. Discrete Lowest Weight Representations $\hat{\mathcal{D}}_j^+$ : These representations are labeled by a real number $j$. $j$ is related to the second Casimir via $c_2 = \frac{1}{2}\{J_0^+, J_0^-\} - (J_0^z)^2 = -j(j-1)$. One starts with a state $|j,j\rangle$ obeying

$$J_{n>0}^{\pm,z}|j,j\rangle = 0, \quad J_0^-|j,j\rangle = 0, \quad J_0^z|j,j\rangle = j|j,j\rangle, \tag{6.56}$$

   and then acts with the remaining operators of the algebra $\{J_{n<0}^{\pm,z}, J_0^+\}$ to obtain the entire representation.

2. Continuous Lowest Weight Representations $\hat{\mathcal{C}}_j^\alpha$ : These representations are labelled by a real number $s$ with $j = \frac{1}{2} + is$. The second Casimir $c_2 = -j(j-1) = s^2 + \frac{1}{4}$. One starts with a state $|s,\alpha,\alpha\rangle$ obeying

$$J_{n>0}^{\pm,z}|j,\alpha,\alpha\rangle = 0, \quad J_0^z|j,\alpha,\alpha\rangle = \alpha|j,\alpha,\alpha\rangle, \tag{6.57}$$

   and acts with the remaining operators of the algebra $\{J_{n<0}^{\pm,z}, J_0^\pm\}$ to obtain the entire representation. Evidently, one may restrict $0 \leq \alpha < 1$

Now, notice that both the $SL(2,R)$ and $SU(2)$ models have a 'spectral flow' symmetry. The transformations

$$\begin{aligned} J_n^z &\to J_n^z + \frac{kw}{2}\delta_{n,0}, \quad K_n^z \to K_n^z + \frac{kw}{2}\delta_{n,0} \\ J_n^\pm &\to J_{n\mp w}^\pm, \quad K_n^\pm \to K_{n\pm w}^\pm, \\ L_n^{SL(2,R)} &\to L_n^{SL(2,R)} - wJ_n^z - \frac{kw^2}{4}\delta_{n,0}, \quad L_n^{SU(2)} \to L_n^{SU(2)} + wK_n^z + \frac{kw^2}{4}\delta_{n,0} \end{aligned} \tag{6.58}$$

preserve the algebra (6.53) and (6.55).[6] For the $SU(2)$ algebra, at level $k$, spectral flow by an odd number of units maps us from a representation of lowest weight $j$ to a representation of lowest weight $\frac{k}{2} - j$. Spectral flow by an even number of units maps us back to the representation of lowest weight $j$. However, for the $SL(2,R)$ algebra, spectral flow generically produces a new representation that is not a lowest weight representation at all! We denote these spectrally flowed representations by $\hat{\mathcal{D}}_j^{w,+}$ and $\hat{\mathcal{C}}_j^{w,\alpha}$. It was explained first in [131] and later in [113, 114] that a consistent Hilbert space of bosonic strings propagating in $AdS_3$ is formed by considering all $\hat{\mathcal{D}}_j^{w,+} \otimes \hat{\bar{\mathcal{D}}}_j^{w,+}$ with $\frac{1}{2} < j < \frac{k-1}{2}$ and all $\hat{\mathcal{C}}_{\frac{1}{2}+is}^{w,\alpha} \otimes \hat{\bar{\mathcal{C}}}_{\frac{1}{2}+is}^{w,\alpha}$. Note that the value of $j$ and $w$ on the right and left have to be the same.

## 6.4.2 Linking Classical Solutions to Quantum States

We need to identify which subsector of the spectrum above corresponds to the solutions discussed in Section 6.3.2. Recall that an analysis of supersymmetry allowed us to completely solve the right-moving sector. This, in turn, also determined the monodromy of the left-moving sector, since the left and right moving parts of the solution are constrained to have the same monodromy. The link between classical

[6] In this chapter, we will think of spectral flow as an operation on states that leaves the operators themselves unchanged

solutions and quantum states goes through the monodromy. The key result that we need was proved by Chu et. al. in [127] drawing on the study of [132]. For other studies examining the canonical formalism applied to WZW models, see [133] and references therein.

Recall that the phase space of the WZW model consists of all classical solutions to the action and these are of the form (6.33). The conjugacy class of the monodromy is a well defined function on phase space. Canonical quantization promotes this function to an operator. The authors of [127] considered the $SU(2)$ model. Conjugacy classes of $SU(2)$ are labelled by a single real number $0 \leq \nu < \pi$ with corresponding group element $e^{i\nu\sigma_3}$. In [127], it was shown that states in the representation $\mathcal{L}^j$ with $0 < j < \frac{k}{2}$ were eigenstates of the operator $\nu$ with eigenvalue

$$\nu = \frac{2j+1}{k+2}\pi. \quad [\mathrm{SU}(2)] \tag{6.59}$$

The analysis of [127] is rather intricate but it is not hard to understand the semi-classical origins of formula (6.59). The affine primary of a lowest weight representation, and other states obtained by acting on it with the zero-modes of the algebra, are the states in the representation that have the lowest conformal weight. Hence, we can derive a semi-classical version of formula (6.59) by considering all solutions with a given monodromy and minimizing their conformal weight. Consider, the right moving part of a classical solution of the $SU(2)$ WZW model, which we parameterize as in (6.30)

$$g_2(x^-) = e^{-i\frac{\phi_1(x^-)-\phi_2(x^-)}{2}\sigma_3} e^{i\zeta(x^-)\sigma_2} e^{-i\frac{\phi_1(x^-)+\phi_2(x^-)}{2}\sigma_3}, \tag{6.60}$$

with the boundary condition:

$$g_2(x^- + 2\pi) = e^{i\sigma_3\nu} g_2(x^-). \tag{6.61}$$

We can obtain the currents of this group element using the formulae in (6.35). We find that the zero-modes of the stress energy tensor and $K^z$ current are given by

$$\begin{aligned} L_0^{\mathrm{SU(2)}} &= \frac{1}{2\pi k}\int_0^{2\pi} \left(\cos^2\zeta(\phi_1')^2 + \sin^2\zeta(\phi_2')^2 + (\zeta')^2\right) dx^-, \\ K_0^z &= \frac{k}{2\pi}\int \left(\cos^2\zeta\phi_1' + \sin^2\zeta\phi_2'\right) dx^-. \end{aligned} \tag{6.62}$$

If we minimize the conformal weight in (6.62) subject to the boundary condition (6.61) then we find that the minimum is reached at:

$$\begin{aligned} \zeta &= \text{constant}, \\ \phi_1' = -\phi_2' &= \frac{\nu}{2\pi}x^-, \\ L_0^{\mathrm{SU(2)}} &= \frac{\nu^2}{(4\pi)^2 k}. \end{aligned} \tag{6.63}$$

If we now use the fact that an affine primary of weight $j$ has conformal weight, $\frac{j(j+1)}{k+2}$, we find the semi-classical relation

$$\nu \sim \frac{2j\pi}{k}. \tag{6.64}$$

where the $\sim$ indicates that this relation is semi-classical. Quantum fluctuations will modify this relation to the exact equation (6.59). The formula (6.63) is valid as long as $\nu \leq \pi$ (otherwise, it is shorter to go around the sphere the 'other way'), which is consistent with the fact that the lowest weight affine primaries of the $SU(2)$ affine algebra are capped at $j = \frac{k}{2}$.

To see the significance of the constant value of $\zeta$ in (6.63), we calculate on this solution:

$$K_0^z = \cos(2\zeta)\frac{k\nu}{2\pi} = \cos(2\zeta)j. \tag{6.65}$$

The possible values of $K_0^z$ for the lowest conformal weight of (6.63) range from $[-j, +j]$. This is what we expect since all states in the $SU(2)$ representation built by acting with the *zero-modes* of the affine algebra on the affine primary have the same conformal weight. They are distinguished by their eigenvalues under $K_0^z$ and these eigenvalues can range from $-j \ldots j$ for an affine primary of weight $j$. The highest value of $j$ is what corresponds to the affine primary, as we defined it above, and this is obtained at $\zeta = 0$.

Now, we notice a remarkable fact. At this value of $\zeta$, the solution (6.63) is exactly the form that the right-moving sector, in the zero winding sector, takes in (6.46). This suggests that the solutions (6.46), for $\tilde{J}_0^z < \frac{k}{2}$ correspond to states that, on the right moving side, are affine primaries of $SU(2)$. The solutions with $\tilde{J}_0^z > \frac{k}{2}$ can always be obtained from these solutions by means of the spectral flow operation (6.48). Hence, the solutions with $\tilde{J}_0^z > \frac{k}{2}$ correspond to states that are obtained by spectrally flowing an $SU(2)$ affine primary using (5.56).

We can repeat the semi-classical analysis above for the $SL(2,R)$ affine algebra. Consider a curve in $SL(2,R)$ parameterized by:

$$g_1(x^-) = e^{i\frac{t(x^-)+\theta(x^-)}{2}\sigma_2} e^{\rho(x^-)\sigma_3} e^{i\frac{t(x^-)-\theta(x^-)}{2}\sigma_2}. \tag{6.66}$$

As we explained above, $SL(2,R)$ has three types of conjugacy classes. However, the solutions of (6.46) have a monodromy that belongs to an elliptic conjugacy class. Hence, we will consider the boundary condition:

$$g_1(x^- + 2\pi) = e^{i\sigma_2\nu} g_1(x^-). \tag{6.67}$$

This time, we have:

$$\begin{aligned} L_0 &= \frac{1}{2\pi k}\int_0^{2\pi} (-\cosh^2\rho t'^2 + \sinh^2\rho\theta'^2 + \rho'^2)\, dx^-, \\ J_0^z &= \frac{k}{2\pi}\int_0^{2\pi} (\cosh^2\rho t' - \sinh^2\rho\theta'^2)\, dx^-. \end{aligned} \tag{6.68}$$

The group parameterization (6.66) admits curves that wind around the 'time' direction but restricting to the zero winding sector, we would find that the solution that minimizes the conformal weight is:

$$\begin{aligned} \rho &= \text{constant}, \\ t' = -\theta' &= \frac{\nu}{2\pi} x^-. \end{aligned} \tag{6.69}$$

This solution has conformal weight, and $J_0^z$ eigenvalue:

$$L_0^{\mathrm{SL(2,R)}} = -\frac{\nu^2}{4\pi^2 k}, \quad J_0^z = \cosh 2\rho \frac{k\nu}{2\pi}. \tag{6.70}$$

The value of $L_0^{\mathrm{SL(2,R)}}$ above corresponds to the lowest conformal weight possible in a *discrete* unflowed representation $\hat{\mathcal{D}}^{0,j}$ with

$$\nu \sim \frac{2j\pi}{k}. \tag{6.71}$$

Repeating the argument of [127] for the $SL(2,R)$ affine algebra yields the quantum result:

$$\nu = \frac{2j-1}{k-2}\pi. \quad [\mathrm{SL(2,R)}] \tag{6.72}$$

As above, the formula (6.69) is valid for $j \leq \frac{k}{2}$ which tells us that we should consider the discrete unflowed representations $\hat{\mathcal{D}}^{0,j}$ only for $j < \frac{k}{2}$. The value of $J_0^z$ in (6.70), can range from $j \ldots \infty$ which is exactly what we expect. The affine primary itself, corresponds to the lowest possible value of $j$ which corresponds to $\rho = 0$ in (6.69). At this value of $\rho$, the solution (6.69) becomes identical to the right-moving solution of (6.46). Hence, we conjecture that the solutions (6.46) correspond, on the right-moving side, to affine primaries of discrete representation of the $SL(2,R)$ affine algebra or to spectral flows of these.

As we mentioned above, there are two other types of conjugacy classes of $SL(2,R)$. The hyperbolic conjugacy classes, in particular, correspond to solutions that have monodromy $e^{s\sigma_3}$. The minimum conformal weight for classical solutions with this boundary condition is $L_0^{\mathrm{SL(2,R)}} = \frac{s^2}{4\pi^2 k}$. Hence, solutions with this monodromy correspond to states in continuous representations. The solutions of (6.46), when $J_0^z = \frac{k}{2}$ are then at the bottom of a continuum i.e we can reach the continuum by moving infitesimally away from supersymmetry. We will have more to say on this below.

We are now in a position to identify the solutions of (6.46) with $\frac{1}{4}$ BPS states in spacetime. Recall that we are looking for states of the form $|\text{anything}\rangle|\text{chiral primary}\rangle$. The classical solutions of (6.46) also have this form. They have a very special structure on the right-moving side and an arbitrary solution on the left moving side. It is then natural to conjecture the following

1. Left (Right) movers on the worldsheet give rise to left (right) movers in spacetime.

2. A chiral primary in spacetime is constructed either (a) by taking the affine primary, of a discrete $SL(2,R)$ representation $\hat{\mathcal{D}}^j$ and combining it with an affine primary of the $SU(2)$ representation $\mathcal{L}^j$ [7] or (b) by spectrally flowing a state of this form by $w$ units.

3. The arbitrary left-moving side of (6.46) is subject to global constraints from the right-moving side. Semi-classically, we see that this left-moving state must

[7] In the exact analysis of Section 6.5, we find that this construction must be modified slightly. We need to combine the affine primary of $\hat{\mathcal{D}}^{j+1}$ with the affine primary of $\mathcal{L}^j$ and dress the state with fermion zero modes

belong to the sector $\hat{\mathcal{D}}^{j,w} \times \mathcal{L}^{j'} \times H^0(\mathcal{M}_{p,q})$ of the sigma model on $SL(2,R) \times SU(2) \times \mathcal{M}_{p,q}$. Here, $j' = j$, if $w$ is even and $j' = \frac{k}{2} - j$ otherwise. Of course, we need to impose the left-moving physical state conditions as well.

4. It appears that at special values of the charges, when the right-moving chiral primary has $j = \frac{kw}{2}$ (recall that according to point (2) above all such right-moving chiral primaries in spacetime are related by spectral flow on the worldsheet) we obtain states that are the bottom of continua.

### 6.4.3 Semi-Classical Analysis of Supersymmetric States

We will now verify the conjecture above by checking that the states described above do indeed obey the physical state conditions and BPS relation and also discuss, in more detail, the structure of the arbitrary states that appear on the left-moving side. Classically, we need to impose the constraints (6.28). Quantum mechanically, we will demand that physical states $|a\rangle$ satisfy $L_n|a\rangle = 0$ and we will mod out by spurious states of the form $|a\rangle = L_{-n}|b\rangle$. What about the mass-shell condition? In passing from classical solutions to quantum states, since we are interested *only* in the spectrum, we have the freedom to choose normal ordering constants. This issue is discussed in some more detail in the next section. In this subsection, since we are in a regime where all charges are large compared to 1, we will not be too precise about this and work with a semi-classical mass-shell condition $L_0|a\rangle \sim 0$.

First, consider the construction of chiral-primaries in the zero winding sector. Consider a right moving state $|c^{j,0}\rangle$ that is an affine primary of $\mathcal{L}^j$ and an affine primary of $\hat{\mathcal{D}}^{0,j}$ and in the ground state in $\mathcal{M}_{p,q}$. Then,

$$\begin{aligned}
&\tilde{L}_0|c^j\rangle = (\tilde{L}_0^{\mathrm{SL(2,R)}} + \tilde{L}_0^{\mathrm{SU(2)}})|c^{j,0}\rangle \sim (-\frac{j^2}{k} + \frac{j^2}{k})|c^{j,0}\rangle = 0, \\
&\tilde{L}_n|c\rangle = (\tilde{L}_n^{\mathrm{SL(2,R)}} + \tilde{L}_n^{\mathrm{SU(2)}})|c^{j,0}\rangle = 0, \\
&(\tilde{J}_0^z - \tilde{K}_0^z)|c^{j,0}\rangle = (j-j)|c^{j,0}\rangle = 0.
\end{aligned} \tag{6.73}$$

So, $|c\rangle$ obeys the physical state and supersymmetry conditions. This state cannot be written as a conformal descendant on the worldsheet. Hence it is not spurious. So, it gives us a good description of a spacetime chiral primary. Semi-classically, it appears that $j$ runs over all the values $0 \leq j < \frac{k}{2}$ in half-integral steps. The exact analysis of the next section shows us that we actually obtain a series where $j$ runs over $\frac{1}{2}, 1, \ldots \frac{k}{2} - \frac{1}{2}$.

Now, notice that, given a state that satisfies the physical state and supersymmetry conditions above, the transformations (5.56) take us to another state that also satisfies these conditions. So the state $|c^{j,w}\rangle$ obtained by *simultaneously* spectral flowing $|c^{j,0}\rangle$ by $w$ units in both $SL(2,R)$ and $SU(2)$ is also a good spacetime chiral primary. Note that this process of spectral flow merely extends the series above, from $\frac{1}{2} \ldots \frac{k}{2} - \frac{1}{2}$ to $\frac{kw}{2} + \frac{1}{2} \ldots \frac{k(w+1)}{2} - \frac{1}{2}$ .

This leaves behind gaps at the values $\frac{kw}{2}$. This is exactly the value of $J_0^z$, where the monodromy of the solutions (6.46) becomes 1. It is also the charge assignment for which we explained, in section 6.2, that classical probe brane solutions could escape to infinity. It is tempting to believe then, that at these values of $J_0^z$ then, the chiral primaries lie at the bottom of a continuum. We will discuss this further in a moment.

Continuing with our discussion of discrete states, let us denote the state on the left-moving side as $|a\rangle$ (for 'arbitrary'). The global constraints of the spectrum described above, mean that that

$$|a\rangle \in \hat{\mathcal{D}}^{j,w} \times \mathcal{L}^{\bar{j}(w)} \times H^0(\mathcal{M}_{p,q}), \tag{6.74}$$

where $\bar{j}(w) = j$ if $w$ is even and $\frac{k}{2} - j$ if w is odd. In addition, we must impose the physical state conditions

$$\begin{aligned} L_n|a\rangle &= 0, \\ L_0|a\rangle &\sim 0. \end{aligned} \tag{6.75}$$

To write an energy formula for $|a\rangle$, it is convenient to consider the state $|a^{-w}\rangle$ obtained by spectral flowing $|a\rangle$ by $-w$ units. This takes us to the zero-winding sector in $SL(2,R)$ and to the representation $\mathcal{L}^j$ in $SU(2)$. Note, that the physical state condition implies $L_{n>0}|a^{-w}\rangle = 0$. Now, $|a^{-w}\rangle$ may be indexed by its level in $SL(2,R)$, $N$,[8] its level in $SU(2)$,$h_2$, its level in the internal CFT on $\mathcal{M}_{p,q}$, $h_{int}$ and its $J_0^z$ eigenvalue, $j+Q$ and its $K_0^z$ eigenvalue $j+P$. $Q$ can be negative because, for example, we can act with $J^-_{-1}$ on the lowest weight state, but we have the constraint that $Q \geq -N$. $P$ can be negative too, because we can act with $K_0^-$ on the lowest weight state.

So, the mass shell condition for $|a\rangle$ then reads:

$$\begin{aligned} L_0|a\rangle &= \left( \frac{-(j+\frac{kw}{2})^2}{k} - wQ + \frac{(j+\frac{kw}{2})^2}{k} + wP + N + h_2 + h_{\text{int}} \right) |a\rangle = 0 \\ &\Rightarrow Q = P + \frac{N + h_2 + h_{\text{int}}}{w}. \end{aligned} \tag{6.76}$$

[8] By 'level' here, we mean the oscillator level which is the difference in the conformal weight of the state and the conformal weight of the zero mode. For example, a state $|\Omega\rangle \in \hat{\mathcal{D}}^{j,w}$ has level: $(L_0^{\text{SL(2,R)}} + \frac{j(j-1)}{k-2})|\Omega\rangle \equiv N|\Omega\rangle$

Finally, we may write the spacetime charges of the state $|a\rangle|c^{w,j}\rangle$ as

$$\begin{aligned}
E &= J_0^z + \tilde{J}_0^z = (j + \frac{kw}{2}) + (j + Q + \frac{kw}{2}), \\
&= 2j + kw + P + \frac{N + h_2 + h_{\text{int}}}{w}, \\
L &= J_0^z - \tilde{J}_0^z = P + \frac{N + h_2 + h_{\text{int}}}{w}, \\
J_1 &= K_0^z + \tilde{K}_0^z = 2j + P + kw, \\
J_2 &= K_0^z - \tilde{K}_0^z = P.
\end{aligned} \tag{6.77}$$

The degeneracy of states with a given value of $h_2, P$ and $h_{\text{int}}$ is given to us by the partition functions for the $SU(2)$ WZW model and the internal CFT. There remains the issue of the degeneracy of,non spurious, states with a given value of $N, Q$ that obey the physical state conditions (6.75). If we are interested only in the degeneracy and not in the actual construction of physical states, the formula for the spacetime partition function in the next section tells us to proceed as follows:

1. Consider the affine primary of $\hat{\mathcal{D}}^j$ and act on it with the oscillator modes $J^+_{n\leq 0}, J^-_{n<0}$, never acting with $J^z_{n<0}$. Let us call this set $\mathcal{Z}$. Now, consider the states obtained by spectral flow of the states in $\mathcal{Z}$ by $w$ units. Call this set $\mathcal{Z}^w$.

2. Consider the states in the tensor product $\mathcal{Z}^w \times \mathcal{L}^{\bar{j}(w)} \times H^0(\mathcal{M}_{p,q})$. Decompose the character of this tensor product into representations of the Virasoro algebra [6], pick out Virasoro primaries and impose the mass shell condition $L_0 \sim 0$.

The procedure above is valid for all states that lie in discrete representations and we have argued this is true for almost all assignments of charges. We now turn to the case where $\tilde{J}_0^z = \frac{E-L}{2} = \frac{kw}{2}$. As we explained in section 6.2, at this value of the charge, the giant graviton solution can go off to infinity. This infinite volume factor means that, upon quantization, the probability that a probe brane with these charges will be found at any finite value of $\rho$ is infitesimally small. Hence, to quantize solutions that have this value of $E - L$ we may simplify the formulae of Table 5.2 by taking the $\rho \to \infty$ limit. Furthermore, since at infinity, such a solution must wrap the $\theta$ direction to have finite energy, we set $\theta' = w$. The remaining dynamical variables are $\rho, \zeta, \phi_1, \phi_2, z^a$ and we have:

$$\begin{aligned} &P_\rho = -\frac{k}{2\pi}\rho', \quad P_\zeta = \frac{-k}{2\pi}\zeta', \quad P_{z^a} = \frac{-k}{2\pi}\left[g_{ab}^{int} z^{b'}\right], \\ &\tilde{P}_{\phi_1} = \frac{k}{2\pi}\left[-\cos^2(\zeta)(\phi_1' - w/2) + \sin^2(\zeta)(\phi_2' - w/2) + \frac{w}{2}\right] = \frac{kw}{2\pi} - \tilde{P}_{\phi_2}. \end{aligned} \tag{6.78}$$

All the complicated couplings between the different degrees of freedom have vanished in the $\rho \to \infty$ limit! Quantizing the $z^a$ and their conjugate momenta leads, as we explained in Section 6.2, to the left-moving sector of the non-linear sigma model on $\mathcal{M}_{p,q}$. Quantizing $\zeta, \phi_1, \phi_2$ leads as one may expect to the left moving sector of the $SU(2)$ WZW model at level $k$. The $\rho$ theory gives rise to a $U(1)$ theory. In terms of this $U(1) \times SU(2)$ theory, the spacetime energy and angular momentum are given by:

$$E = kw + \frac{N_\rho + h_2 + h_{\text{int}}}{w}, \quad L = \frac{N_\rho + h_2 + h_{\text{int}}}{w}. \tag{6.79}$$

where $h_2, h_{\text{int}}$ are as above and $N_\rho$ is the level of the $U(1)$ theory.

In fact this $U(1)\times SU(2)$ theory is nothing but the theory of long-strings studied in [79].[9] We refer the reader to that paper for details but recount two salient conclusions. First, the $U(1) \times SU(2)$ theory admits a $N = 4$ supersymmetric extension To obtain this we need to improve the $U(1)$ model with a linear dilaton term that increases the central charge of the supersymmetric $U(1)\times SU(2)$ model to $6(k-pq)$. Now, with the

[9] A closely related theory was studied in [117, 116, 115]

$N = 4$ supersymmetric sigma model on $\mathcal{M}_{p,q}$ we have a $N = 4$ superconformal theory on the worldsheet and it is not hard to show from here that the entire superconformal symmetry carries over from the worldsheet to spacetime via (6.79).

We also notice that the $N = 4$ theory in the NS sector we have obtained above may be obtained by performing spectral flow in *spacetime* (to be distinguished from spectral flow on the worldsheet, that we have been discussing), on the theory of long-strings in the background of the zero mass BTZ black hole (the Poincare patch of $AdS_3$ with a circle identification) that was discussed in [120]. However, the quantization of strings in that background did not yield any of the discrete states that we have found in global AdS. This provides further evidence for the argument made in [77] that the Poincare patch is *not* the correct background dual to the Ramond sector of the boundary theory.

We can also obtain the energy formula above using the analysis of states in continuous representations in [113] and this analysis shows us that they lie at the bottom of a continuum. The measure for continuous representations was worked out in [114] and since the supersymmetric states above correspond to a particular point (the bottom) and not to a range in the continuum, they are actually of measure zero.

Nevertheless, semi-classically we seem to have a complete story. For generic charges, $\frac{1}{4}$ BPS states occur in discrete representations with an energy given by (6.77) and at special values of the charges, where the classical solutions can escape to infinity, they appear at the bottom of a continuum with energy given by (6.79). In the exact analysis of the D-string carried out in Section 6.5, this story is almost completely borne out except for the puzzling fact that the measure for continuous representations vanishes in a neighbourhood of the point where we expect to find supersymmetric states. This leads to missing chiral primaries at special values of charges. We discuss this issue and the implications of the observation above for the elliptic genus in the next section.

### Half-BPS States

We have provided a semi-classical description of $\frac{1}{4}$ BPS states above. We now discuss $\frac{1}{2}$ BPS states in spacetime. These are of the form $|\text{chiral primary}\rangle|\text{chiral primary}\rangle$. We will denote them by $|j_L, j_R\rangle$ where $j_L, j_R$ are the R-charge values on the left and the right.

It is easy to construct such states on the worldsheet. They are merely, states of the form $|c^{j,w}\rangle|c^{j,w}\rangle$. For concreteness, consider a D-string so that $k = Q_5$. The discussion above tells us that semi-classically, we should expect a chain of such states. First, we consider $w = 0$ and all possible values of $j$. This leads to chiral states in spacetime of the form $|j, j\rangle$ with values of $\frac{1}{2} \leq j \leq \frac{Q_5-1}{2}$. Now, we spectral flow these states to obtain states of the form $|j + \frac{Q_5 w}{2}, j + \frac{Q_5 w}{2}\rangle$. There are gaps in the chiral primary spectrum at $j = \frac{Q_5 w}{2}$ because at these values the chiral-primaries lie in the continuum as we discussed above. The 'exclusion principle' tells us that we must restrict to $w \leq Q_1$. This semi-classical picture captures all the essential features of the exact analysis that we perform in the next section. The inclusion of fermionic zero-modes gives a degeneracy to each element of this chain. Furthermore, it is also possible to have $j_L = j_R \pm \frac{1}{2}$. The exact spectrum is worked out in the next section.

Somewhat more curiously, we seem to get a copy of this series of $\frac{1}{2}$ BPS states from each kind of $(p, q)$ probe. However, this is not a surprise when we recall that the probe solutions corresponding to chiral-chiral primaries are geodesics that do not know anything of the internal structure of the probe. Hence, to obtain the correct spectrum of half-BPS states on the boundary, we should count the chiral-primaries only once and not repeatedly. The simplest way to avoid over-counting is to consider the chiral primaries obtained from the single and multi-particle states of the D-string. In the next section, we show how the $\frac{1}{2}$ BPS spectrum of the boundary theory may be reproduced this way. We could also use a different probe although chiral primaries obtained from a multi-particle state of the D-string may be the same as chiral-primaries obtained from a single particle state of a more complicated probe.

## 6.5 Exact Analysis of the D-string

In this section, we will analyze the exact spacetime partition function for the D-string. When the string coupling is large in the D-brane picture, $Q_1 >> \frac{vQ_5}{g^2}$, we can perform a S-duality to obtain a weakly coupled F-NS5 system. The motion of D-strings in global $AdS_3 \times S^3$ with RR fluxes but no NS fluxes is dual to the propagation of F-strings in $AdS_3 \times S^3$ with NS fluxes but no RR fluxes. This system has been widely studied. For some early studies of string propagation on $AdS_3$ and its relation to the AdS/CFT correspondence, see [126, 134, 135]. In this section, we will rely heavily on the papers [113, 114]. Please also refer to these papers for a review of the early literature on string theory on $AdS_3$. For later studies, see [136, 137, 138]. The supersymmetric extension of the partition function of [114] that we will use here was studied in [139].

For ease of presentation, we will work in the background $AdS_3 \times S^3 \times T^4$. The calculations we perform here may be easily repeated for $K3$, and none of the results we will obtain here are affected. Our plan of attack will be to generalize the spacetime partition function of the bosonic string calculated in [114] to the superstring. By taking various limits of this partition function, we will then obtain expressions for the degeneracies of $\frac{1}{4}$ BPS states and $\frac{1}{2}$ BPS states. We will also discuss the elliptic genus. This section contains several messy manipulations with infinite products that we relegate to Appendix 6.A leaving only the results for the main text.

We start with the formula for the spacetime partition function of the bosonic string, derived in [114]. In terms of the $AdS$ energy, $E$, and angular momentum $L$, the levels of the boundary CFT are given by [57]

$$L_0^b = \frac{E+L}{2}, \quad \tilde{L}_0^b = \frac{E-L}{2}. \tag{6.80}$$

Then

$$\begin{aligned} Z(\beta,\bar{\beta}) &= \mathrm{Tr}_{\mathrm{single-particles}} e^{-\beta L_0^b - \bar{\beta}\tilde{L}_0^b} \\ &= \frac{b(Q_5-2)^{1/2}}{8\pi} \int_0^\infty \frac{d\tau_2}{\tau_2^{3/2}} \int_{-1/2}^{1/2} d\tau_1 e^{4\pi\tau_2(1-\frac{1}{4(Q_5-2)})} Z_{\mathrm{SU(2)}}(q,\bar{q}) Z_{\mathrm{int}}(q,\bar{q}) \\ &\times \frac{e^{-(Q_5-2)b^2/4\pi\tau_2}}{|\sinh(\beta/2)|^2} \left| \prod_{n=1}^\infty \frac{1-q^n}{(1-e^{\hat{\beta}}q^n)(1-e^{-\beta}q^n)} \right|^2, \end{aligned} \tag{6.81}$$

where, $q = e^{2\pi i\tau}, \tau = \tau_1 + i\tau_2$. The formula above is valid when $b \equiv \mathrm{Re}(\beta) = \mathrm{Re}(\bar{\beta}) > 0$. Here, $Z_{\mathrm{int/SU(2)}} = Tr_{T^4/SU(2)}(q^{L_0^b}\bar{q}^{\tilde{L}_0^b})$. Notice the absence of the usual $\frac{c}{24}$ shift. This zero point energy has already been taken into account in (6.81).

Let us expand some of the terms in the formula above. First, we consider the $SU(2)$ partition function. We will denote the level of this model by $k-2$. At this level, the character of the representation $\mathcal{L}^j$, built on an affine primary with weight $j$

is given by[112]:

$$\chi_j^{k-2}(\tau,\rho) = tr(q^{L_0^{\mathrm{SU(2)}}} z^{K_0^z}) = \frac{q^{\frac{1}{8}-\frac{1}{4k}} \sum_{n\in\mathbb{Z}} q^{\frac{(j+\frac{1}{2}+kn)^2}{k}} \left(z^{j+\frac{1}{2}+kn} - z^{-(j+\frac{1}{2}+kn)}\right)}{i\theta_1(\rho,\tau)}, \quad (6.82)$$

where, $z = e^{2\pi i \rho}$.

Naively, one may think that the formula above has a pole of order 1 when $z = q^w$ where $w \in \mathbb{Z}$. However, this is not the case, because the numerator also vanishes for this assignment of chemical potentials. The formula above is valid whenever $|q| < 1$. The partition function of the bosonic $SU(2)$ WZW model, at level $k-2$ is given by

$$Z^{k-2}_{SU(2)-bosonic}(\tau,\rho) = \sum_{j=0}^{k/2-1} |\chi_j^{k-2}(\tau,\rho)|^2. \quad (6.83)$$

The spacetime $SU(2)$ charges are measured by the worldsheet $SU(2)$ charges, and in the notation of Table 5.2, the spacetime $SU(2)$ angular momenta $J_L, J_R$ are given by

$$J_R = \frac{J_1 + J_2}{2} = \tilde{K}_0^z, \quad J_L = \frac{J_1 - J_2}{2} = K_0^z. \quad (6.84)$$

Next, we need to include fermions and generalize the expression to the superstring. As explained in [126] and references therein, the addition of fermions in the $SL(2,R)$ and $SU(2)$ WZW models is simple. We obtain decoupled fermionic and bosonic WZW models, except that the level of the bosonic $SL(2,R)$ model is shifted to $Q_5+2$ and the level of the bosonic $SU(2)$ model is shifted to $Q_5-2$. To generalize expression (6.81) then, we merely need to add in the worldsheet partition function for the fermions and alter the zero-point energies and levels appropriately. This is done in Appendix 6.A which also discusses the sum over R-NS sectors and the GSO projection. The final ingredient we need is the worldsheet partition function of the internal $T^4$.

Putting all of this together we find that the full partition function of a F-string propagating in $AdS_3 \times S^3 \times T^4$ with $Q_5$ units of NS flux is given by

$$\begin{aligned} Z(\beta,\bar\beta,\rho,\bar\rho) &= \mathrm{Tr}_{\mathrm{single-particles}} e^{-\beta L_0^b - \bar\beta \tilde L_0^b - \rho J_L - \bar\rho J_R} \\ &= \frac{bQ_5^{\frac{1}{2}}}{2\pi}\int \frac{d\tau_2}{\tau_2^{\frac{3}{2}}}\int_{\frac{-1}{2}}^{\frac{1}{2}} d\tau_1 \left[\sum_{j=0}^{\frac{Q_5}{2}-1}\left|\left(\frac{\sum_{n\in\mathbb{Z}} q^{\frac{(j+\frac{1}{2}+Q_5 n)^2}{Q_5}}\left(z^{j+\frac{1}{2}+Q_5 n} - z^{-(j+\frac{1}{2}+Q_5 n)}\right)}{\theta_1(\frac{i\rho}{2\pi},\tau)}\right)\right|^2 \right. \\ &\left. \times \left(\sum_{\Gamma^{4,4}} q^{p_L^2}\bar q^{p_R^2}\right)\left|\frac{\left(\theta_2(i\frac{\beta+\rho}{4\pi})\theta_2(i\frac{\beta-\rho}{4\pi})\right)^2}{\theta_1(\frac{i\beta}{2\pi},\tau)\eta(\tau)^6}\right|^2 e^{-Q_5\frac{b^2}{4\pi\tau_2}}\right]. \end{aligned} \tag{6.85}$$

The theta functions can be decoded by looking in Appendix 6.B. The contribution of the zero mode momenta and winding on the $T^4$ is contained in the sum over the lattice $\Gamma^{4,4}$. However, as in [140], we will focus on states that carry 0 charge under $p_L$ and $p_R$.

Some of the symmetry of the spacetime theory is already visible in (6.85). The zero modes of the $\theta$ functions generate the global supergroup $SU(1,1|2)$. The zero modes of the four theta functions in the numerator correspond to the action of 8 left moving and 8 right moving supercharges. The zero modes of the theta functions in the denominator correspond to the action of $K_0^-$ and $J_0^+$.

It is not hard to repeat the calculation of the partition function above for $K3$; if we work at a value of $K3$ moduli where $K3$ is just $T^4/Z_2$, then we merely need to modify the $T^4$ partition function above by adding in twisted sectors and projecting onto invariant states. This will add 3 more terms to the last line of (6.85), lead to a different spectrum of chiral primaries below and give a finite result for the elliptic genus. However, none of our conclusions or puzzles below are affected.

### 6.5.1 $\frac{1}{4}$ BPS Partition Function

The $\frac{1}{4}$ BPS partition function for the D-string is obtained from the formula (6.85) by taking the limit $\bar{\beta} \to \infty, \bar{\rho} \to -\infty$ keeping $\bar{\beta} + \bar{\rho} = -\mu$ finite. It is shown in Appendix 6.A that in this limit we can ignore all right-moving oscillator contributions. It is then possible to do the integral over $\tau_2$ in (6.85) and the remaining integral over $\tau_1$ then just provides a level matching condition.

Let us define the function $f$ by:

$$\frac{\theta_2(\frac{u-v}{2},\tau)^2\theta_2(\frac{u+v}{2},\tau)^2}{-i\theta_1(u,\tau)\eta(\tau)^6}\chi_j^{Q_5-2}(\tau,v) = \sum_{Q,P,h} f_j(Q,P,h)e^{2\pi i u Q}e^{2\pi i v(P+j+\frac{1}{2})}e^{2\pi i\tau\{\frac{j(j+1)}{Q_5}+h\}}, \tag{6.86}$$

where we expand the left hand side in the regime where $0 < \mathrm{Im}(u) < \mathrm{Im}(\tau)$. Note, that at any given power of $q = e^{2\pi i\tau}$, the expansion in powers of $z = e^{2\pi i v}$ terminates after a finite number of terms. In terms of this function $f$, the $\frac{1}{4}$ BPS partition function is given by a remarkably simple formula.

$$\begin{aligned} Z_{\frac{1}{4}}(\beta,\rho,\bar{\mu}) &= 4\cosh^2\frac{\bar{\mu}}{4}\sum_{w\geq 0}\sum_{j=0}^{\frac{Q_5}{2}-1} e^{-\bar{\mu}(j+\frac{Q_5 w}{2}+\frac{1}{2})} \\ &\times \sum_{\substack{w(Q-P)\\-h=0}} f_j(Q,P,h)e^{-\beta(Q+j+\frac{1}{2}+\frac{Q_5 w}{2})-\rho(P+j+\frac{1}{2}+\frac{Q_5 w}{2})}. \end{aligned} \tag{6.87}$$

Comparing this with (6.77), if we redefine $j \to j+\frac{1}{2}$ we find exact agreement with the formulae for the charges given there. Thus we see, as promised, that the semi-classical formula in section 6.4 has given us an exact answer with all factors of 1 correct, at least for the D-string. It is tempting to conjecture that this is also the case for $(p,q)$ strings.

Note that, for $w > 0$, we may replace $Q$ in the second line by $Q = P + \frac{h}{w}$. For

$w = 0$, the sum runs over terms that have $h = 0$. These terms come from the zero modes in the theta functions in (2.38) and give us the graviton multiplets described in [122].

Although we have written the sum (6.87) over all positive $w$, the exclusion principle proposed in [125] along the lines of [93, 122] instructs us to cut off this sum at $w = Q_1$.

### 6.5.2 $\frac{1}{2}$ BPS Partition Function

Now we will try and obtain the spectrum of chiral-chiral states. To do this, in addition to the limit above, we need to take the limit $\beta \to \infty$, $\rho \to -\infty$, keeping $\beta + \rho = -\mu$ finite. It is shown in Appendix 6.A that in this limit, we can ignore all contributions from the theta functions except for the zero modes of $\theta_2(i\frac{\beta+\rho}{4\pi})$ in (6.85). The character for the chiral primaries then becomes:

$$\begin{aligned}
Z_{\frac{1}{2}}(\mu, \bar{\mu}) =& \mathrm{tr}_{\mathrm{chiral-primaries}} e^{\mu K_0^z + \bar{\mu}\tilde{K}_0^z} \\
= \lim & \frac{bQ_5^{\frac{1}{2}}}{2\pi} \int \frac{d\tau_2}{\tau_2^{\frac{3}{2}}} \int_{\frac{-1}{2}}^{\frac{1}{2}} d\tau_1 \\
& e^{\frac{-Q_5 b^2}{4\pi\tau_2}} \sum_{j=0}^{Q_5/2-1} \left| q^{(j+\frac{1}{2}+Q_5 n)^2/Q_5} \left( z^{j+\frac{1}{2}+Q_5 n} - z^{-(j+\frac{1}{2}+Q_5 n)} \right) \left( 2\cosh^2\frac{\mu}{4} \right) \right|^2 \\
= & \sum_{n\in\mathbb{Z}^+} \sum_{j=0}^{Q_5/2-1} \left( 2\left( t^{\frac{1}{2}} + \frac{1}{t^{\frac{1}{2}}} + \bar{t}^{\frac{1}{2}} + \frac{1}{\bar{t}^{\frac{1}{2}}} \right) + 4 + \frac{t^{\frac{1}{2}}}{\bar{t}^{\frac{1}{2}}} + \frac{\bar{t}^{\frac{1}{2}}}{t^{\frac{1}{2}}} + t^{\frac{1}{2}}\bar{t}^{\frac{1}{2}} + \frac{1}{t^{\frac{1}{2}}\bar{t}^{\frac{1}{2}}} \right) (t\bar{t})^{j+\frac{Q_5 n}{2}+\frac{1}{2}},
\end{aligned} \tag{6.88}$$

where $t = e^{\mu}$. This is in agreement with [141]. This analysis can easily be repeated for $K3$ to obtain a spectrum in agreement with [123, 122]. If we apply the exclusion principle mentioned above, then the highest power of $t$ that appears above is $\frac{Q_1 Q_5}{2}$.

Notice, though that the chiral primaries that correspond to $j = Q_5/2$ in the series above are not present. We expect this from our semi-classical analysis above. On the boundary, these missing chiral primaries result from the small instanton singularity [79]; in the bulk this phenomenon was first noticed in [125]. There, it was suggested, as we reasoned above, that these missing chiral primaries disappear into the continuum.

Let us examine this hypothesis. In Appendix 6.A we show that chiral primaries can occur in the continuous spectrum if the condition

$$j + Q_5 n + \frac{1}{2} = \frac{Q_5 w}{4} + \frac{1}{w}\left(\frac{s^2}{Q_5} + \frac{(j + Q_5 n + \frac{1}{2})^2}{Q_5}\right) \tag{6.89}$$

is met with $w$ being some integer. This can only happen if:

$$\begin{aligned} s &= 0, \\ j + Q_5 n + \frac{1}{2} &= \frac{Q_5 w}{2}. \end{aligned} \tag{6.90}$$

This appears promising before we realize that this condition cannot be met because the sum over $j$ runs from $0 \ldots \frac{Q_5}{2} - 1$. We discuss this issue further in Section 6.5.4.

### 6.5.3 Elliptic Genus

We now turn to a study of the elliptic genus. The elliptic genus is defined as

$$E(\beta, \rho) = \mathrm{tr}\{e^{-\beta L_0^b - \rho J_L - \bar{\beta}(\tilde{L}_0^b - J_R)}(-1)^{2J_R}\}. \tag{6.91}$$

The chemical potential $\bar{\beta}$ is purely formal; the elliptic genus is independent of this parameter.

For $T^4$ the elliptic genus vanishes due to fermion zero modes. Although, we could repeat this calculation for $K3$ to obtain a finite elliptic genus, we will instead consider the quantity:

$$E_2(\beta, \rho) = \mathrm{tr}\{e^{-\beta L_0^b - \rho J_L - \bar{\beta}(\tilde{L}_0^b - J_R)}(-1)^{2J_R} J_R^2\}. \tag{6.92}$$

This quantity was defined and studied in [140], specifically to study BPS states in toroidal string theory. The trace is taken only over states that have no $U(1)^4$ charge. In the formula of (6.85) this instructs us to drop the sum over $\Gamma^{4,4}$. We then find:

$$\begin{aligned} E_2(\beta,\rho) &= \left.\frac{\partial^2 Z_{\frac{1}{4}}(\beta,\rho,\bar{\mu})}{\partial\bar{\mu}^2}\right|_{\bar{\mu}=2\pi i} \\ &= \frac{1}{2}\sum_{w\geq 0}\sum_{j=0}^{\frac{Q_5}{2}-1}\sum_{\substack{w(Q-P)\\-h=0}}(-1)^{2j} f_j(Q,P,h)e^{-\beta(Q+j+\frac{1}{2}+\frac{Q_5 w}{2})-\rho(P+j+\frac{1}{2}+\frac{Q_5 w}{2})}. \end{aligned} \tag{6.93}$$

Notice, that several cancellations occur in the expression above because of the term $(-1)^{2j}$ above.

### 6.5.4 Comparison to the Symmetric Product

Before we compare our results for the elliptic genus and the $\frac{1}{2}$ BPS partition function to the symmetric product, let us briefly review some known results. In [123, 122], de Boer found the spectrum of gravitons in $AdS_3\times S^3\times K3$ and organized it into short representations of the relevant AdS supergroup $SU(1,1|2)_L\times SU(1,1|2)_R$. His results may be generalized to $T^4$, and in that case the spectrum of single-particle gravitons described above consists of the $\frac{1}{2}$ BPS states of Section (6.5.2) and their descendants under the generators of this global supergroup. In formula (6.85), the action of these global generators is seen in the zero-modes of the theta functions. Now, two results were obtained in [122] (See, also [142]). First, it was found that the spectrum of chiral-chiral primaries of the symmetric product up to energies $\frac{Q_1Q_5}{2}$ could be found by multi-particling the spectrum of single particle chiral-chiral primaries of supergravity subject to a suitable exclusion principle. Second, with an extension of this exclusion principle, it was found that the elliptic genus of supergravity also agreed with the elliptic genus of the symmetric product till the energy $\frac{Q_1 Q_5}{4}$. For the case of $T^4$ a similar result regarding the modified Index (6.92) was proved in [140].

These results are surprising, because naively one would expect supergravity to be valid till an energy $Q_5$ (assuming $Q_5 < Q_1$), and expect stringy effects to take over beyond that. Indeed, from formula (6.87), we see that the $\frac{1}{4}$ BPS spectrum of the string theory agrees with supergravity till energies of order $Q_5$ (i.e in the zero-winding sector) but disagrees for energies larger than that.

However, the result of section 6.5.2 shows, as was expected from the semi-classical analysis of Section 6.3.4, that the $\frac{1}{2}$ BPS spectrum of the full string theory agrees with the $\frac{1}{2}$ BPS spectrum of supergravity up to an energy $\frac{Q_1 Q_5}{2}$, barring some missing chiral-primaries. Modulo this complication, the calculation of [122] shows us that multi-particling the spectrum of Equation (6.88) with an appropriate exclusion principle at high-energies will reproduce the spectrum of chiral-chiral states of the symmetric product.

The issue of missing chiral-primaries acquires greater urgency in a consideration of the elliptic genus.[10] From (6.93), we see that for left moving conformal weight larger than $\frac{Q_5}{2}$, the elliptic genus contains contributions from $\frac{1}{4}$ BPS states that are not seen in supergravity. Hence, multi-particling this spectrum leads to a mismatch with the elliptic genus of the symmetric product. This, however, does not contradict any theorem because as we have mentioned the boundary theory is singular on this submanifold of moduli space and has a continuum in its spectrum; this invalidates

[10] Here, we are tacitly assuming that we are on $K3$. For $T^4$ where the elliptic genus vanishes, everything in our discussion is valid with "elliptic genus" replaced by the modified Index (6.92)

the Index theorems that protect the elliptic genus [143].

By modular invariance, the high temperature behaviour of the elliptic genus is dominated by the lowest energy supersymmetric states in the spectrum. Since these new $\frac{1}{4}$ BPS contributions appear after an energy gap, their effect on the high temperature behaviour is *exponentially subleading*. So, they do not affect entropy counting calculations. However, it would be interesting to understand the physical interpretation of these subleading contributions in the spirit of [124]. An interesting possibility is that these subleading terms correspond to multi black holes.

As we deform the theory away from this point in moduli space, the continuum must resolve to give rise to new $\frac{1}{4}$ BPS states $|\text{anything}\rangle|\text{chiral primary}\rangle$ with chiral primaries corresponding to $j = \frac{Q_5}{2}$ in the sum (6.88). This is necessary to supply the missing $\frac{1}{2}$ BPS states and the right $\frac{1}{4}$ BPS states to cancel the extra terms in (6.93). Schematically, this happens as follows.

On this submanifold of moduli space, the single particle partition function of string theory may be written as

$$Z(\beta, \bar{\beta}, \rho, \bar{\rho}) = \sum_{h,\bar{h},r,\bar{r}} n(h, \bar{h}, r, \bar{r}) e^{-\beta h - \bar{\beta}\bar{h} - \rho r - \bar{\rho}\bar{r}} + \sum_{r,\bar{r}} \int \rho(h, \bar{h}, r, \bar{r}) e^{-\beta h - \bar{\beta}\bar{h} - \rho r - \bar{\rho}\bar{r}} \, dh d\bar{h}, \tag{6.94}$$

which represents the contributions from both the discrete and continuous representations. We have seen that the second term does not contribute to the $\frac{1}{4}$ BPS partition function because

$$\rho(h, \bar{h}, r, \bar{h}) = 0, \forall h, \bar{h}, r. \tag{6.95}$$

Now, the energy formula (6.89) does allow states with $\bar{r} = \bar{h}$ to exist in continuous representations. The reason the measure above vanishes for these states is that in the $SU(2)$ WZW model at level $Q_5 - 2$, there is no lowest weight representation of weight $\frac{Q_5}{2} - \frac{1}{2}$. In fact, from formula (6.82), we see that

$$\chi^{Q_5-2}_{\frac{Q_5}{2}-\frac{1}{2}}(\tau,\rho) = 0, \forall \tau, \rho. \tag{6.96}$$

The character of a representation may be obtained by symmetrizing the character of the corresponding Verma module over the Weyl group to remove null states [144]. So, loosely speaking we can interpret (6.96) to mean that all states in this representation are null.

As we deform the theory away from this point in moduli space, we can imagine the supersymmetric spectrum changing via a two step process. In the first step, the continuum resolves into discrete states

$$\sum_{r,\bar{r}} \int \rho(h,\bar{h},r,\bar{r}) e^{-\beta h - \bar{\beta}\bar{h} - \rho r - \bar{\rho}\bar{r}}\, dh d\bar{h} \to \sum_{h,\bar{h},r,\bar{r}} n'(h,\bar{h},r,\bar{r}) e^{-\beta h - \bar{\beta}\bar{h} - \rho r - \bar{\rho}\bar{r}}. \tag{6.97}$$

And in the second step $\frac{1}{4}$ BPS discrete states combine into long representations leaving behind a reduced supersymmetric spectrum.

However, as we move away from this point in moduli space by turning on RR fields, we also deform the worldsheet current algebra. Under such a deformation, the RHS of (6.96) may jump from zero. Then, (6.89) tells us that it is possible that

$$n(h,\bar{h},r,\bar{h}) \neq 0, \text{ for } \bar{h} \in \{\frac{Q_5 w}{2}, \frac{Q_5 w}{2} \pm \frac{1}{2}\}, \tag{6.98}$$

where $w$ is a positive integer. These new discrete states could provide the missing chiral-primaries and also pair up with the extra $\frac{1}{4}$ BPS states to remove them from the supersymmetric spectrum. It would be nice to have a more quantitative understanding of this process.

### 6.5.5 Higher Probes

The partition function for the entire theory is obtained by summing, not only over states of the D-string but also over the more complicated $(p, q)$ probes. Now, if we take the action (6.27) with the substitutions (6.29) seriously, and attempt to quantize it like the fundamental string, we are left with a theory that, for a generic $(p, q)$ probe, has too large a central charge. This is not a surprise, because the manipulations that led to (6.29) were classical in nature. A bona-fide analysis of supersymmetric states in these higher probes must start with the worldvolume theory of the D5 brane.

However, the semi-classical analysis of Section 6.2 and the analysis of long-strings in Section 6.4 suggest a possible resolution. In formulae (6.20), (6.26), (6.79) the non-linear sigma model on $\mathcal{M}_{p,q}$ made its appearance. In the bosonic case, it seems possible to generalize the exact analysis of the D-string by simply substituting the bosonic partition function of $\mathcal{M}_{p,q}$ in place of $Z_{\text{int}}$ in formula (6.81), without changing the zero-point energy (the coefficient of $\tau_2$ in the exponent) at all.

To understand this better, consider the following analogy. Say, we are trying to quantize a bosonic string in $d$ dimensions, where $d$ is not necessarily 26. Let us choose light cone gauge, and impose the mass-shell condition $(L_0 - 1)|\Omega\rangle = 0$. This leads to a spectrum that is free of the Lorentz anomaly. At the massless level, we obtain a representation of $SO(d-2)$ and at higher levels the spectrum reorganizes itself into representations of $SO(d-1)$. Of course, we cannot consistently introduce interactions in this theory, but if we are interested *only* in the spectrum, this procedure leads to a sensible result.

For our case, the supersymmetric spectrum can perhaps be obtained by appropriately supersymmetrizing this bosonic spectrum obtained in this manner.

We conclude this section with a speculative possibility. It is possible that if we sum the contributions to the elliptic genus over all the different $(p, q)$ probes, the contributions from all states except for $\frac{1}{2}$ BPS states cancel. To check this possibility, however, we need to be able to exactly quantize the more complicated $(p, q)$ probes. This is a very interesting problem that we leave to future work.

## 6.6 Results

In this chapter, we first developed an alternative approach to classical probe brane solutions in global $AdS_3$, in terms of the 'Polyakov' action. We showed that the canonical structure on the space of $\frac{1}{4}$ BPS brane probes found in [120] was the same as the canonical structure on the solutions (6.46) of the sigma-model (6.27) except that the 'Polyakov' approach also allowed us to identify the classical solutions corresponding to $\frac{1}{2}$ BPS states. We found that these states were described by geodesics that do not see 'stringy' effects even at energies above $Q_1$ and $Q_5$. This explained several facts about the spectrum of $\frac{1}{2}$ BPS states that had, hitherto, been puzzles.

Second, the 'Polyakov' approach allowed us to recast the problem of quantizing these supersymmetric probes as a problem of quantizing the sigma model (6.27) and picking out the physical subsector of the Hilbert space. We followed this procedure and found that, generically, the quantization of $\frac{1}{4}$ BPS brane probes in global $AdS_3 \times S^3 \times K3/T4$ leads to states in discrete representations of the $SL(2, R)$ WZW model with energy, given as a function of charges, by (6.77). Semi-classically, at special values of the charges, the $\frac{1}{4}$ BPS states are at the bottom of a continuum. Quantizing these probes leads to the long strings studied by Seiberg and Witten with energy given as a function of charges by (6.79).

The presence of these discrete states in global AdS is in sharp contrast to the result obtained by quantizing $\frac{1}{4}$ BPS brane probes in the background of the zero mass BTZ black hole (Poincare patch with a circle identification). There, we only obtain states at the bottom of a continuum. So our results here bolster the argument made in [77] that the Poincare patch is not the correct background dual to the Ramond sector of the boundary theory.

Since, the $\frac{1}{4}$ BPS brane probe solutions cease to exist if we turn on the bulk NS-NS fields or theta angle, we concluded that this leads to a jump in the $\frac{1}{4}$ BPS partition function.

By exactly quantizing the D-string we verified the energy formula (6.77). Furthermore, by taking the appropriate limit of the $\frac{1}{4}$ BPS partition function we obtained, in equation (6.88), the spectrum of single particle chiral-chiral primaries of the D-string. Modulo the issue of some 'missing' chiral primaries at special charges (that result from singularities of the boundary theory at this point in moduli space), multi-particling this spectrum reproduces the spectrum of chiral-chiral primaries of the symmetric product. In section 6.5.3, we found that stringy $\frac{1}{4}$ BPS states in discrete representations contribute to the bulk elliptic genus on the special submanifold of moduli space where the background NS-NS fluxes and theta angle are set to zero. This leads to subleading terms in the elliptic genus of the theory on this submanifold of moduli space that are not present in the elliptic genus of the symmetric product. In Section 6.5.4 we showed that as we move away from this special submanifold, the continuum must resolve in a specific way to cancel these additional contributions and supply the missing chiral primaries.

It would be of interest to extend our analysis of $(p, q)$ bound state probes beyond the semi-classical approximation. This is an important direction for future work.

# 6.A Technical Details of the Spacetime Partition Function

In this appendix, we will fill in the details that lead to the results of section 6.5.

## 6.A.1 Partition function

To generalize the bosonic partition function (6.81) we need to add in fermions and the $\beta\gamma$ ghosts, sum over R-NS sectors, impose the GSO projection and explicitly include the partition function of $T^4$.

First, consider the worldsheet partition function for the $SL(2,R), SU(2)$ and $T^4$ fermions and $\beta\gamma$ ghosts. For each of these, we can calculate the quantity:

$$\begin{aligned} Z(a,b)(\beta,\rho,\tau) &= \mathrm{Tr}((-1)^{bF} e^{\rho K^z - \beta J^z + 2\pi i \tau (L_0 - \frac{c}{24})}, \\ \psi(\sigma + 2\pi) &= (-1)^a \psi(\sigma). \end{aligned} \tag{6.99}$$

These partition functions are listed explicitly in the Table below.

| | $Z(0,0)$ | $Z(1,0)$ | $Z(0,1)$ | $Z(1,1)$ |
|---|---|---|---|---|
| SL(2, R) fermions | $\frac{\theta_2(\frac{i\beta}{2\pi},\tau)\theta_2(0,\tau)^{\frac{1}{2}}}{\eta(\tau)^{\frac{3}{2}}}$ | $\frac{\theta_3(\frac{i\beta}{2\pi},\tau)\theta_3(0,\tau)^{\frac{1}{2}}}{\eta(\tau)^{\frac{3}{2}}}$ | $\frac{\theta_1(\frac{i\beta}{2\pi},\tau)\theta_1(0,\tau)^{\frac{1}{2}}}{\eta(\tau)^{\frac{3}{2}}}$ | $\frac{\theta_4(\frac{i\beta}{2\pi},\tau)\theta_4(0,\tau)^{\frac{1}{2}}}{\eta(\tau)^{\frac{3}{2}}}$ |
| SU(2) fermions | $\frac{\theta_2(\frac{i\rho}{2\pi},\tau)\theta_2(0,\tau)^{\frac{1}{2}}}{\eta(\tau)^{\frac{3}{2}}}$ | $\frac{\theta_3(\frac{i\rho}{2\pi},\tau)\theta_3(0,\tau)^{\frac{1}{2}}}{\eta(\tau)^{\frac{3}{2}}}$ | $\frac{\theta_1(\frac{i\rho}{2\pi},\tau)\theta_1(0,\tau)^{\frac{1}{2}}}{\eta(\tau)^{\frac{3}{2}}}$ | $\frac{\theta_4(\frac{i\rho}{2\pi},\tau)\theta_4(0,\tau)^{\frac{1}{2}}}{\eta(\tau)^{\frac{3}{2}}}$ |
| $\mathrm{T}^4$ fermions | $\frac{\theta_2(0,\tau)^2}{\eta(\tau)^2}$ | $\frac{\theta_3(0,\tau)^2}{\eta(\tau)^2}$ | $\frac{\theta_1(0,\tau)^2}{\eta(\tau)^2}$ | $\frac{\theta_4(0,\tau)^2}{\eta(\tau)^2}$ |
| $\beta\gamma$ ghosts | $\frac{\theta_2(0,\tau)}{\eta(\tau)}$ | $\frac{\theta_3(0,\tau)}{\eta(\tau)}$ | $\frac{\theta_1(0,\tau)}{\eta(\tau)}$ | $\frac{\theta_4(0,\tau)}{\eta(\tau)}$ |

(6.100)

Finally, the worldsheet fermionic partition function may be written as

$$
\begin{aligned}
&Z_{\text{fer}}(\beta,\rho,\bar{\beta},\bar{\rho},\tau,\bar{\tau}) \\
&= \left|\frac{\theta_2(\frac{i\beta}{2\pi})\theta_2(\frac{i\rho}{2\pi})\theta_2(0)^2 - \theta_1(\frac{i\beta}{2\pi})\theta_1(\frac{i\rho}{2\pi})\theta_1(0)^2 + \theta_4(\frac{i\beta}{2\pi})\theta_4(\frac{i\rho}{2\pi})\theta_4(0)^2 - \theta_3(\frac{i\beta}{2\pi})\theta_3(\frac{i\rho}{2\pi})\theta_3(0)^2}{\eta(\tau)^6}\right|^2 \\
&= \left|\frac{\theta_2(i\frac{\beta+\rho}{4\pi})^2\theta_2(i\frac{\beta-\rho}{4\pi})^2}{\eta(\tau)^6}\right|^2 .
\end{aligned}
\tag{6.101}
$$

where, in the last step, we have used the Riemann identity. We can think of this as passing from the R-NS formalism to the Green Schwarz formalism.

## 6.A.2 The Integral

### Chiral Primaries

Recall, as explained in [114] that the integral in (6.85) starts by writing

$$
e^{\frac{-k\beta^2}{4\pi\tau_2}} = \frac{-8\pi i}{\beta}\left(\frac{\tau_2}{k}\right)^{\frac{3}{2}} \int_{-\infty}^{\infty} dc\, c e^{\frac{-4\pi\tau_2}{k}c^2+2i\beta c}. \tag{6.102}
$$

Now, notice that if we expand the other $\theta$ functions in (6.85) then, we will get an exponent of the form

$$\begin{aligned}&\frac{-4\pi\tau_2}{k}c^2 + i(\beta+\bar{\beta})c + 2\pi i\bar{\tau}\Big(\frac{(j+k\bar{n}+\frac{1}{2})^2}{k}+\bar{\ell}\Big) - \bar{\rho}(j+k\bar{n}+\frac{1}{2}+\bar{m}_2) - \bar{\beta}\bar{m}_1\\&-\rho(j+kn+\frac{1}{2}+m_2) - \beta m_1 - 2\pi i\tau\Big(\frac{(j+kn+\frac{1}{2})^2}{k}+\ell\Big)\end{aligned} \tag{6.103}$$

Our notation is slightly different from [114]. The terms $\ell, \bar{\ell}, \bar{m}_1, m_1, \bar{m}_2, m2$ merely come from expanding out all the terms in the partition function (6.85) and we will consider them in more detail in a moment.

The integral over $\tau_2$ splits up into winding sectors, with the winding sector $w$ spanning the range $\frac{b}{2\pi w} < \tau_2 < \frac{b}{2\pi(w+1)}$, where as usual $b = \mathrm{Re}(\beta)$. The integral over $c$ picks up poles at:

$$\frac{-c^2}{k} = \frac{(j+k\bar{n}+\frac{1}{2})^2}{k} + \ell. \tag{6.104}$$

In each winding sector, we have the constraint,

$$\frac{kw}{2} < \mathrm{Im}c < \frac{k(w+1)}{2}, \tag{6.105}$$

while the integral over $\tau_1$ yields the level matching condition

$$\frac{(j+kn+\frac{1}{2})^2}{k} + \ell = \frac{(j+k\bar{n}+\frac{1}{2})^2}{k} + \bar{\ell}. \tag{6.106}$$

Consider the anti-holomorphic part of equation (6.103). Doing the integral over $c$ yields the term:

$$-\bar{\beta}\left(\bar{m}_1 + \sqrt{k\Big(\frac{(j+k\bar{n}+\frac{1}{2})^2}{k}+\bar{\ell}\Big)}\right) - \bar{\rho}(\bar{m}_2 + j + k\bar{n} + \frac{1}{2}) \tag{6.107}$$

Now, note that for this term to survive in the limit $\bar{\beta} = -\bar{\rho} + \bar{\mu} \to \infty$ , we need to have:

$$\bar{m_2} + j + k\bar{n} + \frac{1}{2} = \bar{m_1} + \sqrt{k(\frac{(j + k\bar{n} + \frac{1}{2})^2}{k} + \bar{\ell})}. \tag{6.108}$$

We will now show that this can happen, only if in the expansion of the partition function, we include only 'zero-modes' and no 'oscillator modes'. To lighten the notation, define

$$\bar{t} = j + k\bar{n} + \frac{1}{2}, \quad \delta = \bar{m_2} - \bar{m_1}. \tag{6.109}$$

If $\delta \neq 0$ then equation (6.108) has a solution subject to the constraints (6.105) when

$$\frac{kw}{2} < \delta + t = \frac{k\bar{\ell}}{2\delta} + \frac{\delta}{2} < k\frac{w+1}{2}. \tag{6.110}$$

This inequality implies $\delta > 0$ and we will show, that in this case,

$$\bar{\ell} \geq \delta(w+1). \tag{6.111}$$

Hence, the a solution to (6.108) can never be found, except at $\bar{\ell} = 0, \delta = 0$.

Let us write out some of the $\theta$ functions in (6.85) explicitly:

$$\begin{aligned}
&\frac{\theta_2(i\frac{\bar{\beta}-\bar{\rho}}{4\pi},\bar{q})^2}{\theta_1(\frac{i\bar{\beta}}{2\pi},\bar{q})\theta_1(\frac{i\bar{\rho}}{2\pi},\bar{q})} = \frac{(1+e^{\frac{\bar{\rho}-\bar{\beta}}{2}})^2}{(1-e^{-\bar{\beta}})(1-e^{\bar{\rho}})}\prod_{n=1}^{\infty}\frac{(1+\bar{q}^n e^{-\frac{\bar{\beta}-\bar{\rho}}{2}})^2(1+\bar{q}^n e^{+\frac{\bar{\beta}-\bar{\rho}}{2}})^2}{(1-\bar{q}^n e^{-\bar{\beta}})(1-\bar{q}^n e^{\bar{\beta}})(1-\bar{q}^n e^{-\bar{\rho}})(1-\bar{q}^n e^{\bar{\rho}})}\\
&= (-1)^w \prod_{n=0}^{w}\frac{(e^{-\frac{\bar{\beta}-\bar{\rho}}{2}}\bar{q}^{-n}+1)^2}{(e^{-\bar{\beta}}\bar{q}^{-n}-1)(e^{\bar{\rho}}\bar{q}^n-1)}\prod_{n=w+1}^{\infty}\frac{(1+\bar{q}^n e^{\frac{\bar{\beta}-\bar{\rho}}{2}})^2}{(1-\bar{q}^n e^{\bar{\beta}})(1-\bar{q}^n e^{-\bar{\rho}})}\\
&\times\prod_{n=1}^{\infty}\frac{(1+\bar{q}^n e^{-\frac{\bar{\beta}-\bar{\rho}}{2}})^2}{(1-\bar{q}^n e^{-\bar{\beta}})(1-\bar{q}^n e^{\bar{\rho}})}.
\end{aligned} \tag{6.112}$$

The reason we transformed the first line above into the second line is for ease in series expansion. The integral (6.85) has poles when $\tau_2 = \frac{b}{2\pi w}$ so one has to be careful while expanding in powers of $e^{-\bar{\beta}}$. Here, we are in the regime where, $\frac{b}{2\pi(w+1)} < \tau_2 < \frac{b}{2\pi w}$.

So, in the second line above, we can expand all terms of the form $\frac{1}{1-x}$ as $\sum_0^\infty x^n$. Now, notice that for each term, (6.111) holds. The first product which goes from $1\ldots w$ has $\bar{m}_2 < 0, \bar{m}_1 > 0, \bar{\ell} < 0$ but $|\bar{m}_1 - \bar{m}_2| \geq \frac{|\bar{\ell}|}{w+1}$. The second product which goes from $w+1\ldots\infty$, has $\bar{m}_2 > 0, \bar{m}_1 < 0, \bar{\ell} > 0$ but $\bar{m}_2 - \bar{m}_1 \leq \frac{\bar{\ell}}{w+1}$. The third product has $\bar{m}_1 < 0, \bar{m}_2 < 0, \bar{\ell} > 0$, so it also satisfies (6.111). The other important term in (6.85) is $\theta_2(i\frac{\bar{\beta}+\bar{\rho}}{4\pi}, \bar{q})^2$. Every term in the expansion of this theta function has $\delta = 0$. Hence, the only terms that can satisfy (6.108) are the *zero* modes of this theta function that also have $\bar{\ell} = 0$. It is apparent that (6.111) holds for the eta functions in (6.85). To conclude, we need to consider only the zero-modes in $\theta_2(i\frac{\bar{\beta}+\bar{\rho}}{4\pi})$ and we can neglect everything else in the limit $\bar{\beta} = -\bar{\rho} + \bar{\mu} \to \infty$.

A very similar argument works for the contribution from the continuous representations. The contribution of the continuous representations comes from the divergences in the integral (6.85) near $\tau_2 = \frac{b}{2\pi w}$. To analyze these, we replace $\tau$ by its value at the pole everywhere except in the divergent term and then again expand out the partition function. By the argument above, again, we only need to concern ourselves with zero modes. In the limit $\bar{\beta} = -\bar{\rho} \to \infty$, the contribution from this pole vanishes unless:

$$j + k\bar{n} + \frac{1}{2} = \frac{kw}{4} + \frac{1}{w}\left(\frac{s^2}{k} + \frac{(j + k\bar{n} + \frac{1}{2})^2}{k}\right) \tag{6.113}$$

is met. This can only happen if:

$$\begin{aligned} s &= 0, \\ j + k\bar{n} + \frac{1}{2} &= \frac{kw}{2}. \end{aligned} \tag{6.114}$$

However, this condition can never be met because the sum over $j$ runs from $0\ldots\frac{k}{2}-1$. Thus it is precisely the chiral primaries that would have been in the continuum that are missing from our list above

### $\frac{1}{4}$ BPS partition function

$\frac{1}{4}$ BPS states are of the form $|\text{anything}\rangle|\text{chiral primary}\rangle$. The first step is to extract the anti-holomorphic chiral primary from the integral, as detailed above. Then, we merely need to series expand the holomorphic term and pick out the term that satisfied the level matching condition (6.106). They key property we need here is

$$\frac{\theta_2(\frac{u-v}{2}+w\tau)^2\theta_2(\frac{u+v}{2})^2}{-i\theta_1(u+w\tau)\eta(\tau)^6}\chi_j^{Q_5-2}(v-w\tau)$$
$$= \begin{cases} z^{\frac{Q_5 w}{2}}q^{-\frac{Q_5 w^2}{4}}\frac{\theta_2(\frac{u-v}{2})^2\theta_2(\frac{u+v}{2})^2}{-i\theta_1(u)\eta(\tau)^6}\chi_j^{Q_5-2}(v) & \text{w even;} \\ -z^{\frac{Q_5 w}{2}}q^{-\frac{Q_5 w^2}{4}}\frac{\theta_2(\frac{u-v}{2})^2\theta_2(\frac{u+v}{2})^2}{-i\theta_1(u)\eta(\tau)^6}\chi_{\frac{Q_5}{2}-j-1}^{Q_5-2}(v) & \text{w odd,} \end{cases} \tag{6.115}$$

where as usual $z = e^{2\pi i v}, q = e^{2\pi i\tau}$. We can use this to shift the arguments of the $\theta$ function to a regime where (2.38) is applicable. Then (6.87) follows.

### Elliptic Genus

To obtain the elliptic genus, we should take $\bar{\rho} = -\bar{\beta} + 2\pi i$. As we mentioned, the partition function (6.85) vanishes with this substitution due to the zero mode contributions from the $\theta$ functions in the numerator. Evaluating the modified Index (6.92) is equivalent to replacing this term with a constant, which in our normalization is $-\frac{1}{2}$. Apart from this we see that with these chemical potentials, dramatic cancellations occur in formula (6.85). We find

$$E_2(\beta,\rho) \sim \frac{-bQ_5^{\frac{1}{2}}}{2\pi} \int \frac{d\tau_2}{\tau_2^{\frac{3}{2}}} \int_{\frac{-1}{2}}^{\frac{1}{2}} d\tau_1 e^{\frac{-Q_5 b^2}{4\pi\tau_2}} \left( \frac{1}{\theta_1(\frac{i\beta}{2\pi},\tau)} \right.$$
$$\times \frac{4\theta_2(i\frac{\beta+\rho}{4\pi})^2\theta_2(i\frac{\beta-\rho}{4\pi})^2}{\eta(\tau)^6} \times \sum_{j=0}^{\frac{Q_5}{2}-1} \left( \frac{\sum_{n\in\mathbb{Z}} q^{(j+\frac{1}{2}+Q_5 n)^2/Q_5}\left(z^{j+\frac{1}{2}+Q_5 n} - z^{-(j+\frac{1}{2}+Q_5 n)}\right)}{\theta_1(\frac{i\rho}{2\pi},\tau)} \right.$$
$$\left.\left. \times\, (-\frac{1}{2}) \cdot (1+e^{\frac{-\bar\beta+\bar\rho}{2}})^2 \frac{\sum_{m\in\mathbb{Z}} \bar q^{(j+\frac{1}{2}+Q_5 m)^2/Q_5}\left(\bar z^{j+\frac{1}{2}+Q_5 m} - \bar z^{-(j+\frac{1}{2}+Q_5 m)}\right)}{(1-e^{-\bar\beta})(1-e^{\bar\rho})} \right) \right). \tag{6.116}$$

In the last line, we will interpret the zero modes of the theta functions that appear as the action of the global generators of the *spacetime* supergroup $SU(1,1|2)$. The term $(1-e^{-\bar\beta})$ which corresponds to an operator with $\bar h = 1, \bar r = 0$ represents the action of $\bar L^b_{-1}$. The term $(1-e^{\bar\rho})$ represents an operator with $\bar h = 0, \bar r = -1$ and corresponds to the action of $\bar K^-$ the lowering operator of the $SU(2)$ R-symmetry. $\bar K^-$ acts on the term in the numerator $(\bar z^{j+\frac{1}{2}+km} - \bar z^{-(j+\frac{1}{2}+km)})$ to generate a $SU(2)$ representation. The two other terms in the numerator $(1+e^{\frac{-\bar\beta+\bar\rho}{2}})^2$ correspond to fermionic operators with $\bar h = \frac{1}{2}, \bar r = -\frac{1}{2}$. This term represents the action of the two global supercharges that do not annihilate the chiral primary at the head of this representation.

If $m > 0$, this chiral primary is represented by the term $\bar z^{j+\frac{1}{2}+Q_5 m}$ in the $SU(2)$ representation. In this case, the term $\bar z^{-(j+\frac{1}{2}+Q_5 m)}$ represents an anti-chiral primary (The reverse is true for $m < 0$). The surviving global supercharges should annihilate this term. It appears that in the formula (6.85) we need to impose this projection by hand. This is equivalent to dropping the term $z^{-(j+\frac{1}{2}+km)}$ in (6.116) and leads to formula (6.93). The same projection needs to be imposed on the holomorphic term in (6.93) and (6.85). This deserves a better understanding.

## 6.B Theta Functions

Here we list our convention for various theta functions. We define:

$$\theta(a,b)(\nu,\tau) = \sum_{p\in\mathbb{Z}} e^{\pi i\tau(p+\frac{a}{2})^2+2\pi i(\nu+\frac{b}{2})(p+\frac{a}{2})}, \tag{6.117}$$

with the conventions:

$$\begin{aligned}
\theta_1 &= \theta(1,1),\\
\theta_2 &= \theta(1,0),\\
\theta_3 &= \theta(0,0),\\
\theta_4 &= \theta(0,1).
\end{aligned} \tag{6.118}$$

Defining, $q = e^{2\pi i\tau}$ and $z = e^{2\pi i\rho}$ the definitions above lead to the following product formulae [112].

$$\begin{aligned}
\theta_1(\rho,\tau) &= -iz^{\frac{1}{2}}q^{\frac{1}{8}}\prod_{n=1}^{\infty}(1-q^n)\prod_{n=0}^{\infty}(1-zq^{n+1})(1-z^{-1}q^n),\\
\theta_2(\rho,\tau) &= z^{\frac{1}{2}}q^{\frac{1}{8}}\prod_{n=1}^{\infty}(1-q^n)\prod_{n=0}^{\infty}(1+zq^{n+1})(1+z^{-1}q^n),\\
\theta_3(\rho,\tau) &= \prod_{n=1}^{\infty}(1-q^n)\prod_{r\in\mathbb{N}+1/2}(1+zq^r)(1+z^{-1}q^r),\\
\theta_4(\rho,\tau) &= \prod_{n=1}^{\infty}(1-q^n)\prod_{r\in\mathbb{N}+1/2}(1-zq^r)(1-z^{-1}q^r).
\end{aligned} \tag{6.119}$$

We sometimes use the abbreviated notation $\theta(\rho)$ for $\theta(\rho,\tau)$. The $\eta$ function is defined by:

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n). \tag{6.120}$$

# Chapter 7

# Future Directions

We have come to the end of our long journey through supersymmetric partition functions in AdS/CFT. In several examples, we were able to match results on both sides of the AdS/CFT correspondence. However, the work in this thesis also provides some pointers for future work.

The first unsolved problem that, historically, motivated several of the studies in this thesis is to count the entropy of supersymmetric black holes in $AdS_5$. We conjectured that this entropy could be obtained from the classical cohomology of a particular supercharge. Enumerating the states in this cohomology is a well defined, simple to state and tantalizing combinatoric problem. However, it has not yet been solved. Perhaps some additional physical insight, over and above combinatoric prowess, is required for this.

These black holes have another striking property. They have only four charges although, naively, one would expect them to have five. The regularity of the gravity solution provides us with one relation between these charges. What is the origin of this relation in the gauge theory? This is another question on which little progress has been made despite much effort.

One might ambitiously hope that numerical work would answer some of these questions. Indeed, the recent remarkable success in simulating supersymmetric quantum mechanics [70] leads us to hope that, in the not so distant future, $\mathcal{N} = 4$ Yang Mills may be amenable to simulation. This would allow far more stringent tests of the AdS/CFT correspondence than have hitherto been considered.

In the $AdS_3/CFT_2$ example, we are left with a puzzle regarding the elliptic genus. As we discussed, the elliptic genus seems to jump as we move off a special submanifold of parameter space. To understand, in detail, how this happens we need to learn how to couple string theory to RR fluxes.

Finally, in the past few years, some progress has been made in quantizing strings in curved spacetime by listing and quantizing classical supersymmetric solutions. This has led us to some puzzles. For example, in $AdS_5$, two distinct families of classical solutions – corresponding to giant gravitons and dual giant gravitons – are believed to describe the same set of quantum states. Similar questions arise in the calculations of Chapters 5 and 6. It would be interesting to have a better understanding of these issues; this would also tell us the limits of the validity of these semi-classical techniques.

We hope that many of these questions will be answered in the near future!

# Appendix A

# Moving Away from Supersymmetry

## A.1 Introduction

Almost thirty years ago t'Hooft, Polyakov, Migdal and Wilson suggested that large $N$ Yang Mills theory could be recast as a string theory. Electric flux tubes of the confining gauge theory were expected to map to dual fundamental strings. The string coupling constant would map to $\frac{1}{N}$. It was hoped that this picture would then allow us to reach $N = 3$ by means of a $\frac{1}{N}$ expansion.

When the gauge-string duality was finally understood in detail, many aspects of this picture were borne out. However, it was found that supersymmetry was necessary for us to have control over the dual string theory. In this thesis, we have analyzed supersymmetric partition functions in the gauge-gravity correspondence.

A gravity dual to pure large $N$ Yang Mills theory probably exists; the problem is that to understand this dual, we need to understand how to quantize string theory in the presence of non-trivial RR fluxes. However, there are some statements we can make simply from the dual gauge theory.

In this appendix, we will calculate the spectrum of particles in the gravity dual to pure, large $N$ yang mills theory

by decomposing the partition function of the free $SU(N)$ gauge theory on a sphere into a sum over characters of the conformal group (the conformal group is a good symmetry of pure Yang Mills theory precisely at free coupling). According to the AdS/CFT dictionary, representations of the conformal group are in one to one correspondence with particles of the dual string theory; consequently our decomposition determines particle spectrum of interest.

## A.2 Conformal Algebra

Adding dilatations and special conformal transformations to a set of Lorentz generators in 4 dimensions gives the conformal algebra.

$$\begin{aligned}
[D, P_\mu] &= -iP_\mu, \\
[D, K_\mu] &= iK_\mu, \\
[K_\mu, P_\nu] &= 2i(\eta_{\mu\nu}D + M_{\mu\nu}), \\
[M_{\mu\nu}, P_\rho] &= i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu), \\
[M_{\mu\nu}, K_\rho] &= i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu), \\
[M_{\mu\nu}, M_{\rho\sigma}] &= i\left(\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\sigma}M_{\mu\rho}\right).
\end{aligned} \tag{A.1}$$

We are interested in unitary representations of this algebra, where these generators are hermitian. However, it is convenient for the purposes of constructing the representations of this algebra, to choose a basis of generators which satisfies the euclidean conformal algebra [7] and in which the generators are no longer all hermitian. The generators (some $D'$, $P'^{\mu}$, etc.) in this new basis will satisfy the same algebra as above with $\eta_{\mu\nu} \to \delta_{\mu\nu}$. The hermiticity properties of the generators in this basis are:

$$M'^{\dagger} = M',$$

$$D'^{\dagger} = -D',$$

$$P'^{\dagger} = K',$$

$$K'^{\dagger} = P'.$$

From now on, we will use this new set of generators and drop the primes for clarity.

We can extract two sets of $SU(2)$ generators from the Lorentz generators $M_{\mu\nu}$. We define:

$$\begin{aligned} J_1^z &= 1/2(M_{12} + M_{03}), \\ J_2^z &= 1/2(M_{12} - M_{03}), \\ J_1^+ &= 1/2(M_{23} + M_{01} + i(M_{02} - M_{13})), \\ J_2^+ &= 1/2(M_{23} - M_{01} - i(M_{13} + M_{02})), \\ J_1^- &= J_1^{+\dagger}, (\text{A.40}) J_2^- = J_2^{+\dagger}. \end{aligned} \tag{A.2}$$

We will also choose to use an hermitian operator $D'' = iD$ for convenience. We note that the set of generators $M = \{D'', J_1^z, J_1^+, J_1^-, J_2^z, J_2^+, J_2^-\}$ generate the maximal compact subgroup $SO(2) \times SO(4) \subset SO(4,2)$ of the conformal group. We can divide the generators into three sets : $G_0 = \{D, J_1^z, J_2^z\}, G_+ = \{J_1^+, J_2^+, P_\mu\}, G_- =$

$\{K_\mu, J_1^-, J_2^-\}$. With this division, the Lie algebra above has the property that:

$$[g_0, g_+] = g_+^{'},$$

$$[g_0, g_0^{'}] = g_0^{''},$$

$$[g_+, g_-] = g_0,$$

$$[g_0, g_-] = g_-^{'}.$$

where anything with a subscript 0 belongs to linear combinations of operators in $G_0$ and similarly symbols with subscripts $+(-)$ belong to linear combinations of operators in $G_+(G_-)$. These relations make it clear that $G_+$ and $G_-$ act like raising and lowering operators on the charges $G_0$. The operators in $G_0$ commute and we will use these as Cartan generators for the algebra.

It will be convenient to choose linear combinations of the operators in $G_+$ and $G_-$ that diagonalize $G_0$. These combinations are:

$$\begin{aligned} P_w &= P_1 + iP_2, \\ P_{\overline{linew}} &= P_1 - iP_2, \\ P_z &= P_3 + iP_4, \\ P_{\bar{z}} &= P_3 - iP_4. \end{aligned} \tag{A.3}$$

These generators all have well defined weights under the Cartan generators $G_0$.

$$\begin{array}{cccc} & D & J_1 & J_2 \\ J_1^+ & 0 & 1 & 0 \\ J_2^+ & 0 & 0 & 1 \\ P_w & 1 & \frac{1}{2} & \frac{1}{2} \\ P_{\bar{w}} & 1 & -\frac{1}{2} & -\frac{1}{2} \\ P_z & 1 & -\frac{1}{2} & \frac{1}{2} \\ P_{\bar{z}} & 1 & \frac{1}{2} & -\frac{1}{2} \end{array} \tag{A.4}$$

## A.3 Representations of the Conformal Group

Any irreducible representation of the conformal group can be written as some direct sum of representations of the compact subgroup $SO(4) \times SO(2)$:

$$R_{SO(4,2)} = \sum_i \bigoplus R^i_{comp}. \tag{A.5}$$

We are ultimately interested in the occurrence of these representations in the partition function of the conformal Yang-Mills gauge theory quantized on $S^3 \times R$; the hamiltonian of the theory is the dilatation operator $D$. The spectrum of this theory is bounded below and therefore we will be interested in representations of the conformal algebra where the values of the charge $D$ are bounded below. Then there must be some term, $R^\lambda_{comp}$ in the above sum that has the lowest dimension. This term has a highest weight state $|\lambda>$ with weights $\lambda = (d, j_1, j_2)$. The $K^\mu$ operators necessarily annihilate all the states in $R^\lambda_{comp}$ because the $K^\mu$ have negative weight under the operator $D$. If we consider the operation of the $P^\mu$ on this set of states, we generate a whole representation of the conformal algebra with states:

$$[\lambda]^* = R^{\lambda}_{SO(4,2)} = \sum_{n_w n_{\bar{w}} n_z n_{\bar{z}}} P_w^{n_w} P_{\bar{w}}^{n_{\bar{w}}} P_z^{n_z} P_{\bar{z}}^{n_{\bar{z}}} \times R^{\lambda}_{comp}. \tag{A.6}$$

We will denote this set of states by $[d, j_1, j_2]^*$. A careful analysis [5] shows that, barring the trivial case, this representation is unitary if one of the following conditions holds on the highest weight state $|\lambda>$:

$$\begin{aligned} &(i)(\text{A.40})d \geq j_1 + j_2 + 2(\text{A.40})j_1 \neq 0 \quad j_2 \neq 0, \\ &(ii)(\text{A.40})d \geq j_1 + j_2 + 1(\text{A.40})j_1 j_2 = 0. \end{aligned} \tag{A.7}$$

In the case where equality does not hold in these unitarity conditions, the representation is called *long* and all the states produced by the operation of the $P^\mu$ are non-zero.

If equality holds in one of the conditions, then the representation will be a truncated *short* representation in which some of the states listed in (A.6) are 0. A unitary representation is one where we can define a positive definite norm. To find the states that should be absent in a short representation, one can assume that the states in $R^{\lambda}_{comp}$ are normalized in the standard way [7] . Calculating the norm of the states $P^\mu|\lambda>$ will show that when equality holds in the unitarity conditions above, some of these states, say $a_\mu P^\mu|\lambda>$ have norm 0. This should be interpreted as meaning that this state is 0 so that the operator $a_\mu P^\mu$ annihilates $|\lambda>$. The descendants of $a_\mu P^\mu|\lambda>$ then also do not occur in the representation. This last statement needs some care as we will see.

We will list here the possible types of short representations:

- In the generic short representation, $j_1 \neq 0, j_2 \neq 0, d = j_1 + j_2 + 2$ the states of norm 0 occur at level 1. The state $|d+1, j_1 - \frac{1}{2}, j_2 - \frac{1}{2}>$ is *not* found in the representation and its descendants are also not to be found in the representation. The set of all descendants of the state $\lambda' = |d+1, j_1 - \frac{1}{2}, j_2 - \frac{1}{2} >$ is the same as $[d+1, j_1 - \frac{1}{2}, j_2 - \frac{1}{2}]^*$, so that we may write the generic short, irreducible representation as:

$$[d, j_1, j_2] = [d, j_1, j_2]^* - [d+1, j_1 - \frac{1}{2}, j_2 - \frac{1}{2}]^*. \qquad \text{(A.8)}$$

- In the case $j_1 = j_2 = 0$, $d = 1$, the state $|3, 0, 0 >$ is not found. All its descendants are also absent, so we may write the irreducible representation as

$$[1, 0, 0] = [1, 0, 0]^* - [3, 0, 0]^*. \qquad \text{(A.9)}$$

- In the case $j_1 > 0, j_2 = 0$, the state $|d+1, j_1 - \frac{1}{2}, \frac{1}{2} >$ is absent. Note that the weights of this state satisfy the unitarity bound $(i)$ in (2.6). When we delete the states $[d+1, j_1 - \frac{1}{2}, \frac{1}{2}]^*$, we must delete it as a *short* representation, ie. we must not delete the states that do not occur in the short rep $[d+1, j_1 - \frac{1}{2}, \frac{1}{2}]$. We will do a calculation below, using an oscillator representation, showing that this is the correct way to remove the zero norm states in this case. We will have

$$[d, j_1, 0] = [d, j_1, 0]^* - [d+1, j_1 - \frac{1}{2}, \frac{1}{2}]^* + [d+2, j_1 - 1, 0]^*. \qquad \text{(A.10)}$$

## A.4 Characters

The characters for these representations are now easy to compute. First we compute a character of the set of states $[d, j_1, j_2]^*$. We will denote the character of this set of states by a bar on $\chi$:

$$\begin{aligned}
\bar{\chi}_{[d,j_1,j_2]} &= Tr_{[d,j_1,j_2]^*} \exp[iD\theta + iJ_1^z\theta_1 + iJ_2^z\theta_2] \\
&= \sum_{\substack{n_k \geq 0 \\ |m_1| < j_1 \\ |m_2| < j_2}} \langle adjoint| e^{iD\theta + iJ_1^z\theta_1 + iJ_2^z\theta_2} P_w^{n_1} P_{\bar{w}}^{n_2} P_z^{n_3} P_{\bar{z}}^{n_4} |d, m_1, m_2 > \\
&= \frac{\chi_{j_1}^{SU(2)} \chi_{j_2}^{SU(2)} e^{id\theta}}{\prod_{j=1}^4 (1 - \exp[i\vec{\alpha_j} \cdot \vec{\theta}])}
\end{aligned} \tag{A.11}$$

where $\vec{\theta} = (\theta, \theta_1, \theta_2)$, and $\alpha_j$ runs over the 4 generators $P_w, P_{\bar{w}}, P_z, P_{\bar{z}}$ and refers to their weights taken from the table (A.4) , ie $\alpha_1 = (1, 1/2, 1/2)$.

The characters of the possible representations are given by:

1. **Long**, $d > j_1 + j_2 + 2$: $\chi_{[d,j_1,j_2]} = \bar{\chi}_{[d,j_1,j_2]}$

   $(ii)$

2. **Short**, $j_1 = j_2 = 0, d = 1$: $\chi_{[1,0,0]} = \bar{\chi}_{[1,0,0]} - \bar{\chi}_{[3,0,0]}$

3. **Short**, $j_1 > 0, j_2 = 0, d = j_1 + 1$ : $\chi_{[d,j_1,0]} = \bar{\chi}_{[d,j_1,0]} - \bar{\chi}_{[d+1,j_1-1/2,1/2]} + \bar{\chi}_{[d+2,j_1-1,0]}$

4. **Short**, $j_1 > 0, j_2 > 0, d = j_1 + j_2 + 2$ : $\chi_{[d,j_1,j_2]} = \bar{\chi}_{[d,j_1,j_2]} - \bar{\chi}_{[d+1,j_1-1/2,j_2-1/2]}$

We will note shortly that these characters are *not* orthogonal. Nevertheless, they can be used to decompose the spectrum of the Conformal Yang Mills theory we are interested in.

# A.5 Oscillator Construction

Here we will discuss an oscillator construction [145] for the SO(4,2) algebra and use it to confirm the character of the short representations $j_2 = 0$ and $d = j_1 + 1$. The SO(4,2) algebra may be represented by 8 bosonic oscillators $a^I, b^J, a_I$ and $b_J$ $(I, J = 1, 2)$ having the following commutation relations:

$$[a_I, a^J] = \delta_I^J (\text{A.40}) [b_P, b^Q] = \delta_P^Q. \tag{A.12}$$

The generators of the $SO(4,2)$ group are represented as:

$$\begin{gathered} J_1^i = 1/2(\sigma^i)_I^J [a^I a_J - 1/2\delta_J^I a^K a_K], \quad J_2^i = 1/2(\bar{\sigma}^i)_P^Q [b^P b_Q - 1/2\delta_Q^P b^R b_R] \\ D = 1/2(N_a + N_b + 2), (\text{A.40}) P^{IJ} = a^I b^J, (\text{A.40}) K_{IJ} = a_I b_J. \end{gathered} \tag{A.13}$$

We note that a state constructed out of oscillators acting on a vacuum satisfying $a_I|0 >= b_J|0 >= 0$ has weights $(1/2(N_a + N_b + 2), 1/2(n_{a_1} - n_{a_2}), 1/2(n_{b_1} - n_{b_2}))$ under $D, J_1^z, J_2^z$ ($n_{a_1}$ is the number of $a^1$ operators used to create the state and $N_a = n_{a^1} + n_{a^2}$). The unitarity constraints are built into this representation, so we may calculate with it without worrying about states that have norm zero. For example, we may compute the "blind" partition function of the short representation $|\lambda >= (j_1 + 1, j_1, 0)$. We first choose a state with the right weights to act as the primary:

$$(a^2)^{2j_1}|0> . \tag{A.14}$$

Now we can easily generate from this state, a representation of the maximal compact subgroup $SO(4) \times SO(2)$:

$$a^{I_1} a^{I_2} a^{I_3} \ldots a^{I_{2j_1}}|0> . \tag{A.15}$$

There are $2j_1 + 1$ states here, all with dimension $D = j_1 + 1$ as we expect. Now we operate with all possible $P^\mu$:

$$\begin{aligned} Z_{[j_1+1,j_1,0]} = \sum_{n_1,n_2,m_1,m_2} \sum_{I_k} < adjoint|x^D a_1^{n_1} a_2^{n_2} b_1^{m_1} b_2^{m_2} a^{I_1} a^{I_2} a^{I_3} \dots a^{I_{2j_1}}|0 > \\ = \sum_{N=0}^{\infty} x^{N+j_1} N(N+2j_1) \\ = \frac{x^{j+1}(2j_1+1-4j_1x+(2j_1-1)x^2)}{(1-x)^4}. \end{aligned} \quad \text{(A.16)}$$

This agrees with the result in the list of characters above. In the second line, we have used the fact that $n_1 + n_2 = m_1 + m_2$ and that the number of $a$s in (A.15) is $2j_1$. This calculation can easily be repeated with chemical potentials added for the angular momenta.

## A.6 Character Decomposition

Character decomposition integrals are evaluated over the Haar measure of the group in question, in this case $SO(4,2)$. We can reduce these integrals to integrals over the maximal torus of the maximal compact subgroup $SU(2) \times SU(2) \times SO(2)$ using the Weyl integration formula

$$\int_G f(g) d\mu_G = \frac{1}{|W|} \int_T f(t) \prod_{\alpha \in R} (1 - \exp(\alpha(t))) d\mu_T. \quad \text{(A.17)}$$

where $f(g)$ is a function satisfying $f(hgh^{-1}) = f(g)$ so that it only depends on the conjugacy class of $g$, and $d\mu_G$ and $d\mu_T$ are the Haar measures on the group $G$ and the maximal torus $T$. $\alpha \in R$ means the product is over the roots of $SO(4,2)$. Each root corresponds to a generator in table (A.4) , for example the factor corresponding to $K_w$ is $(1 - \exp(-i(\theta + \frac{\theta_1+\theta_2}{2})))$. The constant $|W|$ is the order of the Weyl group in the compact case. In this non-compact case, it will diverge. We nevertheless obtain a useful orthogonality relation below where this constant is not relevant. An integral of characters over the group G becomes:

$$\begin{aligned}\int_G \chi^*_{[d,j_1,j_2]}\chi_{[d',j_1',j_2']}d\mu_G =&\frac{1}{|W|}\int_0^{2\pi}\int_0^{4\pi}\int_0^{4\pi}\chi^*_{[d,j_1,j_2]}(\theta,\theta_1,\theta_2)\chi_{[d',j_1',j_2']}(\theta,\theta_1,\theta_2)\\ &\prod_{\alpha\in R}(1-\exp(i\vec{\alpha}\cdot\vec{\theta}))\frac{d\theta}{2\pi}\frac{d\theta_1}{4\pi}\frac{d\theta_2}{4\pi}.\end{aligned} \tag{A.18}$$

While the characters of the non-compact group $SO(4,2)$ are not orthogonal, the characters of the sets of states $[d,j_1,j_2]^*$ can easily be shown to explicitly satisfy the following orthogonality relation:

$$\frac{1}{4}\int \bar{\chi}^*_{[d,j_1,j_2]}\bar{\chi}_{[d',j_1',j_2']}\prod_{\alpha\in R}(1-\exp(i\vec{\alpha}\cdot\vec{\theta}))d\mu_T = \delta_{d,d'}\delta_{j_1,j_1'}\delta_{j_2,j_2'}. \tag{A.19}$$

This orthogonality is enough for us to decompose the partition function of YM into representations of the conformal group.

In the case of non-compact groups, character decomposition integrals involve some subtleties. Written naively, these integrals have poles. To learn how to deal with these poles, consider the representation $[1,0,0]\times[1,0,0]$. The decomposition of this tensor product by characters will involve integrals like

$$\begin{aligned}&\int(\bar{\chi}_{[1,0,0]})^2\bar{\chi}^*_{[d,j_1,j_2]}d\mu_G =\\ &\int\frac{(\cos j_1\theta_1-\cos(j_1+1)\theta_1)(\cos j_2\theta_2-\cos(j_2+1)\theta_2)\exp[-id\theta]\exp[2i\theta]}{\prod_{\alpha\in P}(1-e^{i\vec{\alpha}\cdot\vec{\theta}})}\frac{d\theta}{2\pi}\frac{d\theta_1}{4\pi}\frac{d\theta_2}{4\pi},\end{aligned} \tag{A.20}$$

where $\alpha\in P$ means product only over the 4 roots corresponding to momentum generators $P_i$ as in (A.11) . It is clear that this integral has singularities along the contour of integration. To resolve this, we deform the contour *inwards.*This is equivalent to *ignoring* the contribution from the boundaries.

To see why this is justified, expand

$$\prod \frac{1}{1 - xq_i} = \sum x^n q_i^n. \tag{A.21}$$

We have introduced new notation here. $x = e^{i\theta}$, $q_i = e^{i\frac{\pm\theta_1 \pm \theta_2}{2}}$ for $i = 1, 2, 3, 4$. Now recalling that $x$ measures the scaling dimension or the energy, we see that that we should add a small imaginary part to $\theta$ which is equivalent to inserting an energy cutoff in the integral.

With this pole prescription, the decomposition yields:

$$\chi_{[1,0,0]*[1,0,0]} = \sum_{d=2}^{\infty} \chi_{[d, \frac{d-2}{2}, \frac{d-2}{2}]}. \tag{A.22}$$

These representations are generically short (barring $[2, 0, 0]$).

We can count the operators in our theory manually to check this result. Using two scalar field representations, the primary operators in the tensor product at the first few levels are:

$$\begin{aligned} &\phi_1\phi_2 \qquad [2, 0, 0], \\ &\phi_2\partial_\mu\phi_1 \qquad [3, \frac{1}{2}, \frac{1}{2}], \\ &\partial_\mu\phi_1\partial_\nu\phi_2 \qquad [4, 1, 1]. \end{aligned} \tag{A.23}$$

which agrees with the decomposition. We will use this same pole prescription in performing the decomposition of the YM theory.

## A.7 The Integral

The single trace partition function of Free Yang Mills on a sphere was calculated in [4, 3]. The result was written as

$$Z[\theta,\theta_1,\theta_2] = -\sum \frac{\phi(k)}{k}\ln(1-z(k\theta,k\theta_1,k\theta_2)), \tag{A.24}$$

where the 'single particle' partition function, z is given by:

$$z = 1 + \frac{(x-x^3)\sum_i q_i + x^4 - 1}{\prod_i(1-xq_i)}. \tag{A.25}$$

We need to decompose this expression as a sum of characters of the conformal group. First,

$$1 - z = \frac{(1-x^2)(x^2-(\sum q_i)x+1)}{\prod_i(1-xq_i)}. \tag{A.26}$$

So, the logarithm in (A.24) will separate the factors here into terms which we will integrate one at a time. Also, we have explicitly

$$\bar{\chi}_{d,j_1,j_2} = \frac{\sin[(j_1+1/2)\theta_1]}{\sin[\frac{\theta_1}{2}]}\frac{\sin[(j_2+1/2)\theta_2]}{\sin[\frac{\theta_2}{2}]}\exp[id\theta]\prod_i \frac{1}{(1-xq_i)}. \tag{A.27}$$

The measure of integration is:

$$dM = 4\sin^2[\frac{\theta_1}{2}]\sin^2[\frac{\theta_2}{2}]\prod_i(x-1/q_i)(x-q_i)\frac{d\,\theta}{2\pi}\frac{d\,\theta_1}{4\pi}\frac{d\,\theta_2}{4\pi}. \tag{A.28}$$

Note that the $\theta_1, \theta_2$ integrals go over $0, 4\pi$.

We will evaluate

$$\int dM Z[\theta,\theta_1,\theta_2]\bar{\chi}^*_{[d,j_1,j_2]}. \tag{A.29}$$

Half of the measure cancels with the denominator of the character. The remaining part of the measure may be written as

$$
\begin{aligned}
&4\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\prod_i (x-q_i) \\
&= 4\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\{(x^4+1) - 4\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}(x^3+x) + 2x^2(\cos\theta_1+\cos\theta_2+1)\}.
\end{aligned}
\tag{A.30}
$$

We will do the integral over the 4 linear factors in the denominator of (A.26) first. The contribution from the partition function $Z$ is

$$
-\sum_{k,i}\frac{\phi(k)}{k}\log\frac{1}{1-x^k q_i^k} = -\sum_{k,i,n}\phi(k)\frac{x^{kn}q_i^{kn}}{kn} = -\sum_{k,n}\phi(k)\frac{4\cos\frac{kn\theta_1}{2}\cos\frac{kn\theta_2}{2}x^{kn}}{kn}.
\tag{A.31}
$$

The integration over $\theta$ picks out coefficients of $x^d$ in the product of (A.31) and the measure (A.30) . The coefficient of $x^d$ in (A.31) is

$$
c(d) = -\sum_{k|d}\frac{\phi(k)}{d}4\cos\frac{d\theta_1}{2}\cos\frac{d\theta_2}{2} = -4\cos\frac{d\theta_1}{2}\cos\frac{d\theta_2}{2}.
\tag{A.32}
$$

Hence, we need to deal with the integral

$$
\begin{aligned}
&\int\Big[c(d)+c(d-4) - 4\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\left(c(d-1)+c(d-3)\right) \\
&\quad + 2\left(\cos\theta_1+\cos\theta_2+1\right)c(d-2)\Big]\left(\cos j_1\theta_1 - \cos\{(j_1+1)\theta_1\}\right) \\
&\quad \times\left(\cos j_2\theta_2 - \cos\{(j_2+1)\theta_2\}\right)\frac{d\,\theta_1}{4\pi}\frac{d\,\theta_2}{4\pi}
\end{aligned}
\tag{A.33}
$$

With $\Delta^a_b = \delta^a_b + \delta^{-a}_b$ the contribution from the factors in the denominator of (A.26) is given by

$$
\begin{aligned}
&I_1[d,j_1,j_2] \\
&= -\Delta^{j_1}_{j_2}\left(\Delta^d_{2j_1} + \Delta^{d-4}_{2j_1} + 2\Delta^{d-2}_{2j_1}\right) + \Delta^{d-1\pm1}_{2j_1}\Delta^{d-1\pm1}_{2j_2} + \Delta^{d-3\pm1}_{2j_1}\Delta^{d-3\pm1}_{2j_2} \\
&- \Delta^{d-2\pm2}_{2j_1}\Delta^{d-2\pm2}_{2j_2}.
\end{aligned}
\tag{A.34}
$$

Next, we consider the $(1-x^2)$ factor in (A.26) .

$$-\log(1-x^{2k}) = \sum \frac{x^{2kn}}{n}. \tag{A.35}$$

This time, for the coefficient of $x^d$, we have

$$\sum_{2k|d} \frac{2\phi(k)}{d} = \begin{cases} 1 & \mathrm{d\,even} \\ 0 & \mathrm{otherwise} \end{cases} \tag{A.36}$$

Substituting this into the main integral, we find that for $d > 4$, we need to integrate

$$\begin{aligned} &\int 4 + 2(\cos\theta_1 + \cos\theta_2 + 1) dM \quad \mathrm{d\,even}, \\ &\int -4\cos\tfrac{\theta_1}{2}\cos\tfrac{\theta_2}{2} dM \qquad \mathrm{d\,odd}. \end{aligned} \tag{A.37}$$

For $d < 4$ the expression above and below should be modified to drop terms that cannot contribute to the pole in $x$.

Define

$$I_0[j_1, j_2] = 4\Delta^0_{j_1}\Delta^0_{j_2} + \Delta^{j_1-1}_0\Delta^{j_2}_0 + \Delta^{j_2-1}_0\Delta^{j_1}_0. \tag{A.38}$$

With $I = I_1 + I_0$, the contribution from the second term is

$$I_2[d, j_1, j_2] = I[d, j_1, j_2] - I[d, j_1, j_2+1] - I[d, j_1+1, j_2] + I[d, j_1+1, j_2+1]. \tag{A.39}$$

Finally we consider the remaining quadratic term in (A.26) . We will call $\sum_i q_i^k = 4\cos\frac{k\theta_1}{2}\cos\frac{k\theta_2}{2} = \alpha_k$, to save space.

$$\begin{aligned} -\log(1-(\alpha_k x^k - x^{2k})) &= \sum_n \frac{(\alpha_k x^k - x^{2k})^n}{n} \\ &= \sum_{p,q} (-1)^q \alpha_k^p x^{(p+2q)k} \frac{1}{p+q} \begin{pmatrix} p+q \\ q \end{pmatrix}. \end{aligned} \tag{A.40}$$

Again we will want to collect the coefficient of $x^d$ here. A term in the sum above contributes to this coefficient only if $p + 2q = d/k$. Also, this expression is summed over $k$ against $\phi(k)/k$. This means we need to consider the sum

$$\sum_{k|d} \sum_{p=0}^{d/k} \frac{\phi(k)}{k} \alpha_k^p (-1)^{(\frac{d}{k}-p)/2} \frac{2}{\frac{d}{k}+p} \begin{pmatrix} \frac{d}{2k} + p/2 \\ p \end{pmatrix}. \tag{A.41}$$

We now look at a generic integral, integrating this term against $\cos\frac{A\theta_1}{2}\cos\frac{B\theta_2}{2}$. All terms occurring in the actual integral of the term in (A.40) may be reduced to this form. Use the identity

$$\int 4^p \left[\cos\frac{k\theta_1}{2}\right]^p \cos\frac{A\theta_1}{2} \left[\cos\frac{k\theta_2}{2}\right]^p \cos\frac{B\theta_2}{2} = \begin{pmatrix} p \\ \frac{1}{2}(p-\frac{A}{k}) \end{pmatrix} \begin{pmatrix} p \\ \frac{1}{2}(p-\frac{B}{k}) \end{pmatrix}. \tag{A.42}$$

To shorten expressions, define $\frac{p}{2} = s$, $\frac{d}{2k} = x$, $\frac{A}{2k} = y$, $\frac{B}{2k} = z$.

This allows us to write the generic integral over the sum in (A.41) as

$$\begin{aligned} &S_1[d,A,B] \\ &= \sum_{k|(d,A,B)} \phi(k) \sum_{s=max(y,z)}^{x} (-1)^{x-s} \begin{pmatrix} s+x \\ 2s \end{pmatrix} \begin{pmatrix} 2s \\ s-y \end{pmatrix} \begin{pmatrix} 2s \\ s-z \end{pmatrix} \frac{1}{k(x+s)}. \end{aligned} \tag{A.43}$$

Now define

$$\begin{aligned} &S_2[d,A,B] = S_1[d,A,B] + S_1[d-4,A,B] \\ &- \sum_{\sigma_1,\sigma_2=\pm 1/2} \Big( S_1[d-1,A+\sigma_1,B+\sigma_2] + S_1[d-3,A+\sigma_1,B+\sigma_2] \Big) \\ &+ \sum_{\rho_1,\rho_2=\pm 1} S_1[d-2,A+\rho_1,B] + S_1[d-2,A,B+\rho_2] \\ &+ 2S_1[d-2,A,B]. \end{aligned} \tag{A.44}$$

Table A.1: Low Dimension Operators in the Pure YM Theory

| **Dimension** | **Operators** | **Conformal Representation Content** |
|---|---|---|
| 1 | no operators | no representations |
| 2 | $F_{\mu\nu}$ | [2,1,0]+[2,0,1] |
| 3 | no primary operators | no representations |
| 4 | $F_{\mu\nu} * F^{\mu\nu}$, $\|F\|^2$ | [4,0,0]+[4,0,0] |
| | $F_{\{\mu\nu}F^{\nu}_{\sigma\}} - \|F\|^2$ | [4,1,1] |
| | $F_{\{\mu\nu}F_{\rho\sigma\}} - F_{\{\mu\nu}F^{\nu}_{\sigma\}} - \|F\|^2$ | [4,2,0]+[4,0,2] |

We can now collect all the terms that appear in the integral over the quadratic term (A.40)

$$I_3[d,j_1,j_2] = S_2[d,j_1,j_2]+S_2[d,j_1+1,j_2+1]-S_2[d,j_1+1,j_2]-S_2[d,j_1,j_2+1]. \quad \text{(A.45)}$$

Collecting the terms contributing, one finds the following enlightening result:

$$\int dMZ[\theta,\theta_1,\theta_2]\bar{\chi}^*_{[d,j_1,j_2]} = I_2[d,j_1,j_2] + I_3[d,j_1,j_2]. \quad \text{(A.46)}$$

where $I_2$ is defined in (A.39) and $I_3$ in (A.20) . As we noted above, for $d < 4$ the expressions get modified.

These sums are prohibitively difficult to evaluate by hand, but may be easily done with a computer.

It is easy to list the operators in the theory at low scaling dimension and we do this in Table A.1.

Now consider large values of $d$. Neglecting the angular variables, we see that, for large values of $d$

$$Z = -\sum \frac{\phi(k)}{k} \ln \frac{(1+x)(x^2-4x+1)}{(1-x)^3} \approx \beta^d x^d. \quad \text{(A.47)}$$

where $\beta = 2+\sqrt{3}$ is the larger root of the quadratic term. This is the characteristic

Table A.2: **Representation Content of Pure YM**

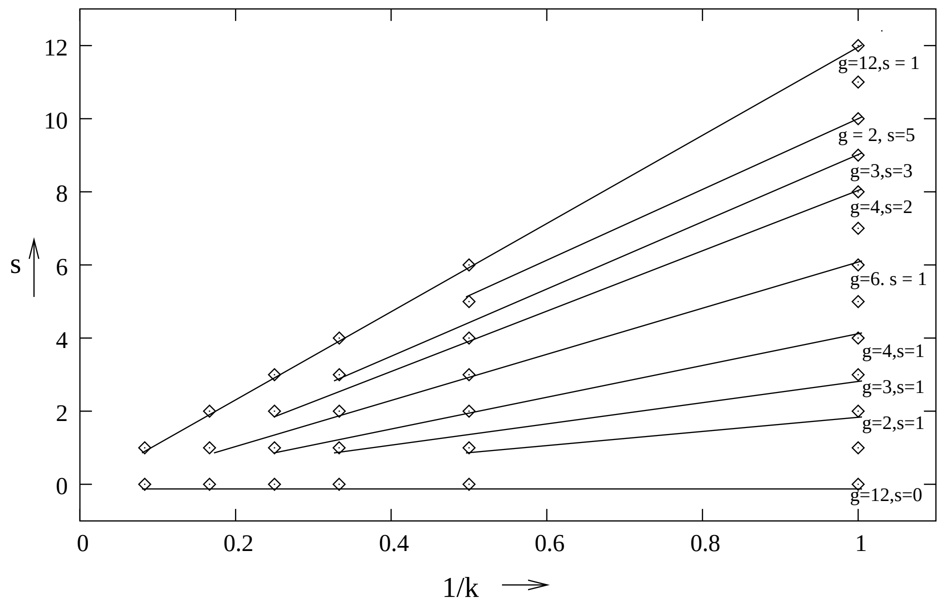
| **Dimension** | **Representations** |
|---|---|
| 2 | [2,0,1] + [2,1,0] |
| 3 | nothing |
| 4 | 2[4,0,0] + [4,0,2] + [4,2,0] + [4,1,1] |
| 5 | [5,3/2,3/2] |
| 6 | 2[6,0,0] + 2[6,0,1] + [6,0,3] + 2 [6,1,0] + 2[6,1,1] |
| | + [6,1,2] + [6,1,3] + [6,2,1] + [6,2,2] + [6,3,0] + [6,3,1] |
| 7 | 4[7,1/2,3/2] + 2[7,1/2,5/2] + 4[7,3/2,1/2] + 4[7,3/2,3/2] |
| | + 2[7,3/2,5/2] + 2[7,5/2,1/2] + 2 [7,5/2,3/2] + [7,5/2,5/2] |
| 8 | 6[8,0,0] + 4[8,0,1] + 5[8,0,2] + [8,0,3] + 2 [8,0,4] |
| | + 4 [8,1,0]+ 10 [8,1,1] + 7[8,1,2] + 5[8,1,3] + [8,1,4] |
| | + 5[8,2,0] + 7 [8,2,1]+ 8 [8,2,2] + 3[8,2,3] + [8,2,4] |
| | + [8,3,0]+ 5[8,3,1] + 3[8,3,2] + [8,3,3]+ 2[8,4,0] |
| | + [8,4,1] + [8,4,2] |
| 9 | 14[9,1/2,1/2] + 20[9,1/2,3/2]+ 15[9,1/2,5/2] + 6[9,1/2,7/2] |
| | + 20[9,3/2,1/2] + 28[9,3/2,3/2] + 18[9,3/2,5/2] + 7[9,3/2,7/2] |
| | + 2 [9,3/2,9/2] + 15[9,5/2,1/2] + 18 [9,5/2,3/2] + 12[9,5/2,5/2] |
| | + 4 [9,5/2,7/2] + 6[9,7/2,1/2] + 7 [9,7/2,3/2] + 4 [9,7/2,5/2] |
| | + [9,7/2,7/2]+ 2[9,9/2,3/2] |

Hagedorn growth in the number of states. It is easy to verify, numerically, that (A.46) does reproduce the right growth in the density of states.

As a final consistency check, it is necessary for (A.46) to sum to an integer for every value of $d, j_1, j_2$; this is not at all apparent from the expression we have. Nevertheless, using some elementary number-theoretic results, we show, in the appendix, that our answer always sums to an integer.

We list here the representation content of the theory up to dimension 9.

Figure A.1: Grouping the Terms




## A.8 Proof of Integral Multiplicities

We now proof that the multiplicities we obtain using the formula above are all integral. Consider (A.43), which has a sum over $k$ and $s$. Schematically, the set of allowed $k, s, x$ values is shown below. In the figure, for each value of $k$, $s$ can range over the values demarcated by the horizontal line at the bottom and the outermost line.

The critical point is to partition this large set of values correctly. We group the set of $k, s$ values in subsets of the kind $S_{x_0,s_0} = \{s, k, x, y, z : x + s = p(x_0 + s_0)\}$ This foliation is indicated on the diagram.

It is clear that in each partition, we have $pk = g$, where $g$ is a constant. Second, we have $\gcd(s_0 + x_0, x_0 - s_0, s_0 - y_0, s_0 - z_0, s_0) = 1$. Since (A.43) can be written as:

$$\frac{\alpha_1}{g(x_0 + s_0)} = \frac{\alpha_2}{2gs_0} = \frac{\alpha_3}{g(x_0 - s_0)} = \frac{\alpha_4}{g(s_0 - y_0)} ...$$

where the $\alpha_i$ are integral, it suffices to show that the sum is divisible by g to show that it is an integer.

This leaves us to prove the following statement:

$$\sum_{k|g} \phi(g/k)(-1)^{k(x_0-s_0)} \begin{pmatrix} (s_0+x_0)k \\ 2ks_0 \end{pmatrix} \begin{pmatrix} 2ks_0 \\ (s_0-y_0)k \end{pmatrix} \begin{pmatrix} 2ks_0 \\ k(s_0-z_0) \end{pmatrix} \mod g = 0. \tag{A.48}$$

For notational simplicity, we consider the simpler statement,

$$\sum_{k|g} (-1)^{n_3 k} \begin{pmatrix} n_1 k \\ n_2 k \end{pmatrix} \phi(g/k) \quad \mod g = 0, \tag{A.49}$$

where $n_i$ are arbitrary integers with $\gcd n_i = 1$. Furthermore, we take $g = p^t$, where $p$ is prime. We take $p \neq 2$ so that $(-1)^{n_3 k}$ has the same sign. The generalization of this proof to generic $g$ is straightforward.

With $t = 1$, our sum is:

$$\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} (t-1) + \begin{pmatrix} n_1 t \\ n_2 t \end{pmatrix} = 0(\mod t). \tag{A.50}$$

where we have used $\begin{pmatrix} n_1 t \\ n_2 t \end{pmatrix} = \begin{pmatrix} n_1 t \\ n_2 t \end{pmatrix}$ mod $t$. Assume the statement is true for $t = n$. For $t = n+1$, our sum is:

$$\sum_{i=0}^{n+1} \begin{pmatrix} n_1 p^i \\ n_2 p^i \end{pmatrix} \phi(p^{n+1-i}) = p \sum_{i=0}^{n} \begin{pmatrix} n_1 p^i \\ n_2 p^i \end{pmatrix} + \begin{pmatrix} n_1 p^{n+1} \\ n_2 p^{n+1} \end{pmatrix} - \begin{pmatrix} n_1 p^n \\ n_2 p^n \end{pmatrix}. \tag{A.51}$$

The first term is divisible by $p^{n+1}$ by hypothesis. With $n_3 = n_1 - n_2$, write the second term as:

$$\begin{pmatrix} n_1 p^n \\ n_2 p^n \end{pmatrix} \left( \frac{n_1 p^{n+1}(n_1 p^{n+1}-1)...(n_1 p^n+1) - n_2 p^{n+1}...(n_2 p^n+1) n_3 p^{n+1}...(n_3 p^n+1)}{n_2 p^{n+1}...n_2 p^n n_3 p^{n+1}..n_3 p^n} \right).$$

We can cancel leading terms divisible by $p^{n+1}$ in the numerator and denominator, but then we notice that subleading terms not divisible by $p^{n+1}$ cancel in the numerator but not in the denominator. So the second term is also divisible by $p^{n+1}$. This proves our result.

## A.9 $\mathcal{N} = 4$ SYM

In principle, it is not difficult to generalize the procedure above to the case of the $\mathcal{N} = 4$ Yang Mills Theory. This theory has an exact superconformal symmetry. Representations of the Superconformal group are labeled by the highest weight under $SO(4) \times SO(2)$ and the *R charges*. These were originally classified in [9]. They were studied in [10, 11] and are discussed in detail in [12] .

It is easy to generalize the partition function by adding in chemical potentials for the R charges. This result can be read off from the appendix in [12]. Similarly, it is simple to generalize the result for the Haar measure and the characters[13].

Unfortunately, short representations in the superconformal case have a rather more intricate structure than in the conformal case and it is not always possible to write them as a difference of two long representations. This complication is not important in the spectrum of single trace operators in the $\mathcal{N} = 4$ theory, because it is known that the only relevant short representations are the $\frac{1}{2}, \frac{1}{2}$ BPS multiplets.

The more serious complication is numerical. Performing the character decomposition involves finding the coefficient of a specified monomial in a given power series expansion. Since we have six chemical potentials in the supersymmetric case, the simple algorithms are $O(d^6)$. Thus, the calculation quickly becomes unfeasible.

In a set of papers [26, 27, 28, 146, 147], Bianchi et. al. conjectured that the spectrum of the free SYM theory may be obtained from the spectrum of type IIB theory on flat space through a specified algorithm. They verified their conjecture using a sieve procedure which allowed them to determine the spectrum up to scaling dimension 10.

Further verification of this conjecture must await either a deeper understanding of their result or the development of more efficient numerical techniques.

## A.10 Summary

Unitary representations of the Conformal Algebra must obey $d \geq j_1 + j_2 + 2$, for $j_1 j_2 \neq 0$ and $d \geq j_1 + j_2 + 1$ otherwise. Depending on whether either of these bounds is saturated, the characters of the conformal group fall into three classes. These are described in Section A.4

The Free Yang Mills theory on a sphere has an exact conformal symmetry. Hence, its partition function may be written as a sum over the characters above. Formally, we have the result

$$Z = \sum N_{d,j_1,j_2} \chi_{d,j_1,j_2}. \tag{A.52}$$

In this note, we performed this decomposition. We find that $N_{d,j_1,j_2}$ is described by (A.46). Our formula demonstrates the correct asymptotics. Moreover, it is possible to prove that it always produces an integer.

# Bibliography


[1] J. M. Maldacena, *The large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231–252, [hep-th/9711200].

[2] E. Witten, *Constraints on Supersymmetry Breaking*, *Nucl. Phys.* **B202** (1982) 253.

[3] B. Sundborg, *The Hagedorn transition, deconfinement and N = 4 SYM theory*, *Nucl. Phys.* **B573** (2000) 349–363, [hep-th/9908001].

[4] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas, and M. Van Raamsdonk, *The Hagedorn / deconfinement phase transition in weakly coupled large N gauge theories*, *Adv. Theor. Math. Phys.* **8** (2004) 603–696, [hep-th/0310285].

[5] G. Mack, *All Unitary Ray Representations of the Conformal Group SU(2,2) with Positive Energy*, *Commun. Math. Phys.* **55** (1977) 1.

[6] A. Barabanschikov, L. Grant, L. L. Huang, and S. Raju, *The spectrum of Yang Mills on a sphere*, *JHEP* **01** (2006) 160, [hep-th/0501063].

[7] S. Minwalla, *Restrictions imposed by superconformal invariance on quantum field theories*, *Adv. Theor. Math. Phys.* **2** (1998) 781–846, [hep-th/9712074].

[8] V. G. Kac, *Lie Superalgebras*, *Adv. Math.* **26** (1977) 8–96.

[9] V. K. Dobrev, G. Mack, V. B. Petkova, S. G. Petrova, and I. T. Todorov, *Harmonic Analysis on the n-Dimensional Lorentz Group and Its Application to Conformal Quantum Field Theory*. Lecture Notes in Physics. Springer-Verlag, Berlin, 1977. 280p.

[10] V. K. Dobrev and V. B. Petkova, *On the Group Theoretical Approach to Extended Conformal Supersymmetry: Classification of Multiplets*, *Lett. Math. Phys.* **9** (1985) 287–298.

[11] V. K. Dobrev and V. B. Petkova, *Group Theoretical Approach to Extended Conformal Supersymmetry: Function Space Realizations and Invariang Differential Operators*, *Fortschr. Phys.* **35** (1987) 537.

[12] F. A. Dolan and H. Osborn, *On short and semi-short representations for four dimensional superconformal symmetry*, *Ann. Phys.* **307** (2003) 41–89, [hep-th/0209056].

[13] V. K. Dobrev, *Characters of the positive energy UIRs of $D = 4$ conformal supersymmetry*, *Phys. Part. Nucl.* **38** (2007) 564–609, [hep-th/0406154].

[14] J. Kinney, J. M. Maldacena, S. Minwalla, and S. Raju, *An index for 4 dimensional super conformal theories*, *Commun. Math. Phys.* **275** (2007) 209–254, [hep-th/0510251].

[15] K. Dasgupta, M. M. Sheikh-Jabbari, and M. Van Raamsdonk, *Protected multiplets of M-theory on a plane wave*, *JHEP* **09** (2002) 021, [hep-th/0207050].

[16] N. Kim and J.-H. Park, *Superalgebra for M-theory on a pp-wave*, *Phys. Rev.* **D66** (2002) 106007, [hep-th/0207061].

[17] H. Lin and J. M. Maldacena, *Fivebranes from gauge theory*, *Phys. Rev.* **D74** (2006) 084014, [hep-th/0509235].

[18] L. Motl, A. Neitzke, and M. M. Sheikh-Jabbari, *Heterotic plane wave matrix models and giant gluons*, *JHEP* **06** (2003) 058, [hep-th/0306051].

[19] H. Liu, *Fine structure of Hagedorn transitions*, hep-th/0408001.

[20] L. Alvarez-Gaume, C. Gomez, H. Liu, and S. Wadia, *Finite temperature effective action, $AdS_5$ black holes, and 1/N expansion*, *Phys. Rev.* **D71** (2005) 124023, [hep-th/0502227].

[21] O. Aharony, J. Marsano, S. Minwalla, and T. Wiseman, *Black hole - black string phase transitions in thermal 1+1 dimensional supersymmetric Yang-Mills theory on a circle*, *Class. Quant. Grav.* **21** (2004) 5169–5192, [hep-th/0406210].

[22] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas, and M. Van Raamsdonk, *A first order deconfinement transition in large N Yang-Mills theory on a small $S^3$*, *Phys. Rev.* **D71** (2005) 125018, [hep-th/0502149].

[23] P. Basu and S. R. Wadia, *R-charged $AdS_5$ black holes and large N unitary matrix models*, *Phys. Rev.* **D73** (2006) 045022, [hep-th/0506203].

[24] O. Aharony *et. al.*, *The phase structure of low dimensional large N gauge theories on tori*, *JHEP* **01** (2006) 140, [hep-th/0508077].

[25] J. F. Morales and H. Samtleben, *Higher spin holography for SYM in d dimensions*, *Phys. Lett.* **B607** (2005) 286–293, [hep-th/0411246].

[26] M. Bianchi, J. F. Morales, and H. Samtleben, *On stringy $AdS_5 \times S^5$ and higher spin holography*, *JHEP* **07** (2003) 062, [hep-th/0305052].

[27] N. Beisert, M. Bianchi, J. F. Morales, and H. Samtleben, *On the spectrum of AdS/CFT beyond supergravity*, *JHEP* **02** (2004) 001, [hep-th/0310292].

[28] N. Beisert, M. Bianchi, J. F. Morales, and H. Samtleben, *Higher spin symmetry and N = 4 SYM*, *JHEP* **07** (2004) 058, [hep-th/0405057].

[29] M. Gunaydin and N. Marcus, *The Spectrum of the s**5 Compactification of the Chiral N=2, D=10 Supergravity and the Unitary Supermultiplets of U(2, 2/4)*, *Class. Quant. Grav.* **2** (1985) L11.

[30] Y. Nakayama, *Index for orbifold quiver gauge theories*, *Phys. Lett.* **B636** (2006) 132–136, [hep-th/0512280].

[31] J. Bhattacharya, S. Bhattacharyya, S. Minwalla, and S. Raju, *Indices for superconformal field theories in 3,5 and 6 dimensions*, 0801.1435.

[32] J. B. Gutowski and H. S. Reall, *Supersymmetric $AdS_5$ black holes*, *JHEP* **02** (2004) 006, [hep-th/0401042].

[33] J. B. Gutowski and H. S. Reall, *General supersymmetric $AdS_5$ black holes*, *JHEP* **04** (2004) 048, [hep-th/0401129].

[34] Z. W. Chong, M. Cvetic, H. Lu, and C. N. Pope, *General non-extremal rotating black holes in minimal five- dimensional gauged supergravity*, *Phys. Rev. Lett.* **95** (2005) 161301, [hep-th/0506029].

[35] A. Strominger and C. Vafa, *Microscopic Origin of the Bekenstein-Hawking Entropy*, *Phys. Lett.* **B379** (1996) 99–104, [hep-th/9601029].

[36] S. Kim, S. Minwalla, S. Raju, and S. Trivedi, "Near horizon geometry of extremal black holes in $ads_5$." unpublished, 2006.

[37] O. Aharony unpublished.

[38] D. Berenstein, *Large N BPS states and emergent quantum gravity*, *JHEP* **01** (2006) 125, [hep-th/0507203].

[39] M. B. Green, J. H. Schwarz, and E. Witten, *SuperString Theory Vol 2: Loop Amplitudes, Anomalies and Phenomenology.* Cambridge Monographs on Mathematical Physics. Cambridge, 1987.

[40] S. S. Gubser, I. R. Klebanov, and A. W. Peet, *Entropy and Temperature of Black 3-Branes*, *Phys. Rev.* **D54** (1996) 3915–3919, [hep-th/9602135].

[41] A. Mikhailov, *Giant gravitons from holomorphic surfaces*, *JHEP* **11** (2000) 027, [hep-th/0010206].

[42] A. Sinha, J. Sonner, and N. V. Suryanarayana, *At the horizon of a supersymmetric $AdS_5$ black hole: Isometries and half-BPS giants*, *JHEP* **01** (2007) 087, [hep-th/0610002].

[43] M. Cvetic *et. al.*, *Embedding AdS black holes in ten and eleven dimensions*, *Nucl. Phys.* **B558** (1999) 96–126, [hep-th/9903214].

[44] F. Cachazo, M. R. Douglas, N. Seiberg, and E. Witten, *Chiral rings and anomalies in supersymmetric gauge theory*, *JHEP* **12** (2002) 071, [hep-th/0211170].

[45] E. D'Hoker, P. Heslop, P. Howe, and A. V. Ryzhov, *Systematics of quarter BPS operators in $N = 4$ SYM*, *JHEP* **04** (2003) 038, [hep-th/0301104].

[46] S. Corley, A. Jevicki, and S. Ramgoolam, *Exact correlators of giant gravitons from dual $N = 4$ SYM theory*, *Adv. Theor. Math. Phys.* **5** (2002) 809–839, [hep-th/0111222].

[47] B. R. Greene, D. R. Morrison, and A. Strominger, *Black hole condensation and the unification of string vacua*, *Nucl. Phys.* **B451** (1995) 109–120, [hep-th/9504145].

[48] G. Arfken, H. Weber, and L. Ruby, *Mathematical Methods for Physicists*, *American Journal of Physics* **64** (1996) 959.

[49] I. Biswas, D. Gaiotto, S. Lahiri, and S. Minwalla, *Supersymmetric states of $N = 4$ Yang-Mills from giant gravitons*, hep-th/0606087.

[50] D. Martelli and J. Sparks, *Dual giant gravitons in Sasaki-Einstein backgrounds*, *Nucl. Phys.* **B759** (2006) 292–319, [hep-th/0608060].

[51] A. Basu and G. Mandal, *Dual giant gravitons in $AdS_m \times Y^n$ (Sasaki-Einstein)*, hep-th/0608093.

[52] R. A. Janik and M. Trzetrzelewski, *Supergravitons from one loop perturbative N=4 SYM*, arXiv:0712.2714.

[53] L. Grant, P. A. Grassi, S. Kim, and S. Minwalla, *Comments on 1/16 BPS Quantum States and Classical Configurations*, arXiv:0803.4183.

[54] M. Berkooz, D. Reichmann, and J. Simon, *A Fermi surface model for large supersymmetric $AdS_5$ black holes*, *JHEP* **01** (2007) 048, [hep-th/0604023].

[55] D. Gaiotto and X. Yin, *Notes on superconformal Chern-Simons-matter theories*, *JHEP* **08** (2007) 056, [0704.3740].

[56] V. K. Dobrev and V. B. Petkova, *All Positive Energy Unitary Irreducible Representations of Extended Conformal Supersymmetry*, *Phys. Lett.* **B162** (1985) 127–132.

[57] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, *Large N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183–386, [hep-th/9905111].

[58] S. Bhattacharyya and S. Minwalla, *Supersymmetric states in M5/M2 CFTs*, *JHEP* **12** (2007) 004, [hep-th/0702069].

[59] M. Gunaydin and N. P. Warner, *Unitary SuperMultiplets of Osp(8/4,R) and the Spectrum of the S(7) Compactification of Eleven-Dimensional Supergravity*, *Nucl. Phys.* **B272** (1986) 99.

[60] B. Biran, A. Casher, F. Englert, M. Rooman, and P. Spindel, *The fluctuating seven-sphere in eleven-dimensional supergravity*, *Physics Letters B* **134** (1984), no. 3-4 179–183.

[61] N. Seiberg, *Notes on theories with 16 supercharges*, *Nucl. Phys. Proc. Suppl.* **67** (1998) 158–171, [hep-th/9705117].

[62] S. Minwalla, *Particles on AdS(4/7) and primary operators on M(2/5) brane worldvolumes*, *JHEP* **10** (1998) 002, [hep-th/9803053].

[63] J. Bagger and N. Lambert, *Comments On Multiple M2-branes*, 0712.3738.

[64] D. J. Gross and E. Witten, *Possible Third Order Phase Transition in the Large N Lattice Gauge Theory*, *Phys. Rev.* **D21** (1980) 446–453.

[65] H. J. Schnitzer, *Confinement / deconfinement transition of large n gauge theories in perturbation theory with n(f) fundamentals: N(f)/n finite*, hep-th/0612099.

[66] V. K. Dobrev, *Positive energy unitary irreducible representations of $D = 6$ conformal supersymmetry*, *J. Phys.* **A35** (2002) 7079–7100, [hep-th/0201076].

[67] M. Gunaydin, P. van Nieuwenhuizen, and N. P. Warner, *General construction of the unitary representations of anti-de sitter superalgebras and the spectrum of the s**4 compactification of eleven-dimensional supergravity*, *Nucl. Phys.* **B255** (1985) 63.

[68] P. Claus, R. Kallosh, and A. Van Proeyen, *M 5-brane and superconformal (0,2) tensor multiplet in 6 dimensions*, *Nucl. Phys.* **B518** (1998) 117–150, [hep-th/9711161].

[69] K. Intriligator and B. Wecht, *Exploring the 4D superconformal zoo*, hep-th/0402084.

[70] K. Anagnostopoulos, M. Hanada, J. Nishimura, and S. Takeuchi, *Monte Carlo Studies of Supersymmetric Matrix Quantum Mechanics with Sixteen Supercharges at Finite Temperature*, *Physical Review Letters* **100** (2008), no. 2 21601, [arXiv:0707.4454].

[71] L. Grant, L. Maoz, J. Marsano, K. Papadodimas, and V. S. Rychkov, *Minisuperspace quantization of 'bubbling AdS' and free fermion droplets*, *JHEP* **08** (2005) 025, [hep-th/0505079].

[72] G. Mandal, *Fermions from half-BPS supergravity*, *JHEP* **08** (2005) 052, [hep-th/0502104].

[73] L. Maoz and V. S. Rychkov, *Geometry quantization from supergravity: The case of 'bubbling AdS'*, *JHEP* **08** (2005) 096, [hep-th/0508059].

[74] V. S. Rychkov, *D1-D5 black hole microstate counting from supergravity*, *JHEP* **01** (2006) 063, [hep-th/0512053].

[75] G. Mandal and N. V. Suryanarayana, *Counting 1/8-BPS dual-giants*, *JHEP* **03** (2007) 031, [hep-th/0606088].

[76] O. Lunin and S. D. Mathur, *Metric of the multiply wound rotating string*, *Nucl. Phys.* **B610** (2001) 49–76, [hep-th/0105136].

[77] O. Lunin, S. D. Mathur, and A. Saxena, *What is the gravity dual of a chiral primary?*, *Nucl. Phys.* **B655** (2003) 185–217, [hep-th/0211292].

[78] O. Lunin, J. M. Maldacena, and L. Maoz, *Gravity solutions for the d1-d5 system with angular momentum*, hep-th/0212210.

[79] N. Seiberg and E. Witten, *The D1/D5 system and singular CFT*, *JHEP* **04** (1999) 017, [hep-th/9903224].

[80] J. McGreevy, L. Susskind, and N. Toumbas, *Invasion of the giant gravitons from anti-de Sitter space*, *JHEP* **06** (2000) 008, [hep-th/0003075].

[81] M. T. Grisaru, R. C. Myers, and O. Tafjord, *SUSY and Goliath*, *JHEP* **08** (2000) 040, [hep-th/0008015].

[82] W.-H. Huang, *Giant gravitons in $AdS_3 \times S^3 \times T^4$ spacetime under Melvin field deformation*, hep-th/0603198.

[83] E. A. Bergshoeff, R. Kallosh, and T. Ortin, *Supersymmetric string waves*, *Phys. Rev.* **D47** (1993) 5444–5452, [hep-th/9212030].

[84] E. Bergshoeff, I. Entrop, and R. Kallosh, *Exact duality in string effective action*, *Phys. Rev.* **D49** (1994) 6663–6673, [hep-th/9401025].

[85] G. Mandal, S. Raju, and M. Smedbäck, *Supersymmetric giant graviton solutions in AdS_ {3}*, *Physical Review D* **77** (2008), no. 4 46011, [arXiv:0709.1168].

[86] J. M. Maldacena and L. Maoz, *De-singularization by rotation*, *JHEP* **12** (2002) 055, [hep-th/0012025].

[87] V. Balasubramanian, J. de Boer, E. Keski-Vakkuri, and S. F. Ross, *Supersymmetric conical defects: Towards a string theoretic description of black hole formation*, *Phys. Rev.* **D64** (2001) 064011, [hep-th/0011217].

[88] E. Bergshoeff and P. K. Townsend, *Super D-branes*, *Nucl. Phys.* **B490** (1997) 145–162, [hep-th/9611173].

[89] E. Bergshoeff, R. Kallosh, T. Ortin, and G. Papadopoulos, *kappa-symmetry, supersymmetry and intersecting branes*, *Nucl. Phys.* **B502** (1997) 149–169, [hep-th/9705040].

[90] M. Eto, Y. Isozumi, M. Nitta, and K. Ohashi, *1/2, 1/4 and 1/8 BPS equations in SUSY Yang-Mills-Higgs systems: Field theoretical brane configurations*, *Nucl. Phys.* **B752** (2006) 140–172, [hep-th/0506257].

[91] J. Polchinski, *String theory. Vol. 2: Superstring theory and beyond*, . Cambridge, UK: Univ. Pr. (1998) 531 p.

[92] O. Lunin and S. D. Mathur, *AdS/CFT duality and the black hole information paradox*, *Nucl. Phys.* **B623** (2002) 342–394, [hep-th/0109154].

[93] J. M. Maldacena and A. Strominger, *$AdS_3$ black holes and a stringy exclusion principle*, *JHEP* **12** (1998) 005, [hep-th/9804085].

[94] J. R. David, G. Mandal, and S. R. Wadia, *Microscopic formulation of black holes in string theory*, *Phys. Rept.* **369** (2002) 549–686, [hep-th/0203048].

[95] M. Banados, *Three-dimensional quantum geometry and black holes*, hep-th/9901148.

[96] A. Strominger, *Black hole entropy from near-horizon microstates*, *JHEP* **02** (1998) 009, [hep-th/9712251].

[97] C. Vafa, *Gas of D-Branes and Hagedorn Density of BPS States*, *Nucl. Phys.* **B463** (1996) 415–419, [hep-th/9511088].

[98] R. Dijkgraaf, *Instanton strings and hyperKaehler geometry*, *Nucl. Phys.* **B543** (1999) 545–571, [hep-th/9810210].

[99] C. Crnkovic and E. Witten, *Covariant description of canonical formalism in geometrical theories*, *Three Hundred Years of Gravitation* (1987) 676–684.

[100] A. Dhar, G. Mandal, S. R. Wadia, and K. P. Yogendran, *D1/D5 system with B-field, noncommutative geometry and the CFT of the Higgs branch*, *Nucl. Phys.* **B575** (2000) 177–194, [hep-th/9910194].

[101] J. M. Maldacena and J. G. Russo, *Large-N limit of non-commutative gauge theories*, *Class. Quant. Grav.* **17** (2000) 1189–1203.

[102] G. Papadopoulos and D. Tsimpis, *The holonomy of IIB supercovariant connection*, *Class. Quant. Grav.* **20** (2003) L253, [hep-th/0307127].

[103] P. Dedecker, *Calcul des variations, formes différentielles et champs geodesiques*, *Geometrie différentielle, Colloq. Intern. du CNRS LII, Strasbourg* (1953) 17–34.

[104] H. Goldschmidt and S. Sternberg, *The Hamilton-Cartan formalism in the calculus of variations*, *Ann. Inst. Fourier* **23** (1973), no. 1 203–267.

[105] J. Kijowski, *A finite-dimensional canonical formalism in the classical field theory*, *Commun. Math. Phys.* **30** (1973) 99–128.

[106] K. Gawedzki and W. Kondracki, *Canonical formalism for the local-type functionals in the classical field theory*, *Rep. Math. Phys* **6** (1974) 465–476.

[107] W. Szczyrba, *A symplectic structure on the set of Einstein metrics*, *Communications in Mathematical Physics* **51** (1976), no. 2 163–182.

[108] P. Garcia, *Reducibility of the symplectic structure of classical fields with gauge symmetry*, *Lecture Notes in Mathematics* **570** (1977).

[109] G. Zuckerman, *Action principles and global geometry*, *Mathematical Aspects of String Theory* **1** (1987) 259–284.

[110] J. Lee and R. M. Wald, *Local symmetries and constraints*, *J. Math. Phys.* **31** (1990) 725–743.

[111] E. Witten, *Nonabelian bosonization in two dimensions*, *Commun. Math. Phys.* **92** (1984) 455–472.

[112] P. Di Francesco, P. Mathieu, and D. Senechal. Springer, New York, USA, 1997.

[113] J. M. Maldacena and H. Ooguri, *Strings in $AdS_3$ and SL(2,R) WZW model. I*, *J. Math. Phys.* **42** (2001) 2929–2960, [hep-th/0001053].

[114] J. M. Maldacena, H. Ooguri, and J. Son, *Strings in $AdS_3$ and the SL(2,R) WZW model. II: Euclidean black hole*, *J. Math. Phys.* **42** (2001) 2961–2977, [hep-th/0005183].

[115] J. Callan, Curtis G., J. A. Harvey, and A. Strominger, *Worldbrane actions for string solitons*, *Nucl. Phys.* **B367** (1991) 60–82.

[116] J. Callan, Curtis G., J. A. Harvey, and A. Strominger, *World sheet approach to heterotic instantons and solitons*, *Nucl. Phys.* **B359** (1991) 611–634.

[117] J. Callan, Curtis G., J. A. Harvey, and A. Strominger, *Supersymmetric string solitons*, hep-th/9112030.

[118] J. de Boer, F. Denef, S. El-Showk, I. Messamah, and D. Van, *Black hole bound states in $AdS_3 \times S^2$*, arXiv:0802.2257.

[119] T. Mansson and B. Sundborg, *Multi-black hole sectors of $AdS_3$ gravity*, *Phys. Rev.* **D65** (2002) 024025, [hep-th/0010083].

[120] G. Mandal, S. Raju, and M. Smedback, *Supersymmetric Giant Graviton Solutions in $AdS_3$*, 0709.1168.

[121] F. Larsen and E. J. Martinec, *U(1) charges and moduli in the D1-D5 system*, *JHEP* **06** (1999) 019, [hep-th/9905064].

[122] J. de Boer, *Large N Elliptic Genus and AdS/CFT Correspondence*, *JHEP* **05** (1999) 017, [hep-th/9812240].

[123] J. de Boer, *Six-dimensional supergravity on $S^3 \times AdS_3$ and 2d conformal field theory*, *Nucl. Phys.* **B548** (1999) 139–166, [hep-th/9806104].

[124] R. Dijkgraaf, J. M. Maldacena, G. W. Moore, and E. P. Verlinde, *A black hole farey tail*, hep-th/0005003.

[125] R. Argurio, A. Giveon, and A. Shomer, *Superstrings on $AdS_3$ and symmetric products*, *JHEP* **12** (2000) 003, [hep-th/0009242].

[126] A. Giveon, D. Kutasov, and N. Seiberg, *Comments on string theory on $AdS_3$*, *Adv. Theor. Math. Phys.* **2** (1998) 733–780, [hep-th/9806194].

[127] M.-f. Chu, P. Goddard, I. Halliday, D. I. Olive, and A. Schwimmer, *Quantization of the Wess-Zumino-Witten model on a circle*, *Phys. Lett.* **B266** (1991) 71–81.

[128] M. R. Gaberdiel and I. Kirsch, *Worldsheet correlators in $AdS_3/CFT_2$*, *JHEP* **04** (2007) 050, [hep-th/0703001].

[129] A. Dabholkar and A. Pakman, *Exact chiral ring of $AdS_3/CFT_2$*, hep-th/0703022.

[130] L. J. Dixon, M. E. Peskin, and J. D. Lykken, *N=2 Superconformal Symmetry and SO(2,1) Current Algebra*, *Nucl. Phys.* **B325** (1989) 329–355.

[131] M. Henningson, S. Hwang, P. Roberts, and B. Sundborg, *Modular invariance of SU(1,1) strings*, *Phys. Lett.* **B267** (1991) 350–355.

[132] G. Felder, K. Gawedzki, and A. Kupiainen, *The Spectrum of Wess-Zumino-Witten Models*, *Nucl. Phys.* **B299** (1988) 355–366.

[133] K. Gawedzki, *Classical origin of quantum group symmetries in Wess-Zumino-Witten conformal field theory*, *Commun. Math. Phys.* **139** (1991) 201–214.

[134] D. Kutasov and N. Seiberg, *More comments on string theory on $AdS_3$*, *JHEP* **04** (1999) 008, [hep-th/9903219].

[135] J. de Boer, H. Ooguri, H. Robins, and J. Tannenhauser, *String theory on $AdS_3$*, *JHEP* **12** (1998) 026, [hep-th/9812046].

[136] A. Giveon and A. Pakman, *More on superstrings in $AdS_3 \times N$*, *JHEP* **03** (2003) 056, [hep-th/0302217].

[137] A. Pakman, *Unitarity of supersymmetric SL(2,R)/U(1) and no-ghost theorem for fermionic strings in $AdS_3 \times N$*, *JHEP* **01** (2003) 077, [hep-th/0301110].

[138] A. Pakman, *BRST quantization of string theory in $AdS_3$*, *JHEP* **06** (2003) 053, [hep-th/0304230].

[139] D. Israel, C. Kounnas, and M. P. Petropoulos, *Superstrings on NS5 backgrounds, deformed $AdS_3$ and holography*, *JHEP* **10** (2003) 028, [hep-th/0306053].

[140] J. M. Maldacena, G. W. Moore, and A. Strominger, *Counting bps black holes in toroidal type ii string theory*, hep-th/9903163.

[141] D. Kutasov, F. Larsen, and R. G. Leigh, *String theory in magnetic monopole backgrounds*, *Nucl. Phys.* **B550** (1999) 183–213, [hep-th/9812027].

[142] P. Kraus and F. Larsen, *Partition functions and elliptic genera from supergravity*, *JHEP* **01** (2007) 002, [hep-th/0607138].

[143] S. Cecotti, P. Fendley, K. A. Intriligator, and C. Vafa, *A New supersymmetric index*, *Nucl. Phys.* **B386** (1992) 405–452, [hep-th/9204102].

[144] J. Fuchs and C. Schweigert, *Symmetries, Lie algebras and representations.* Cambridge University Press New York, NY, USA, 1997.

[145] I. Bars and M. Gunaydin, *Unitary Representations of Noncompact Supergroups*, *Commun. Math. Phys.* **91** (1983) 31.

[146] M. Bianchi, *Higher spin symmetry (breaking) in $N = 4$ SYM and holography*, *Comptes Rendus Physique* **5** (2004) 1091–1099, [hep-th/0409292].

[147] M. Bianchi, *Higher spins and stringy $AdS_5 \times S^5$*, *Fortsch. Phys.* **53** (2005) 665–691, [hep-th/0409304].